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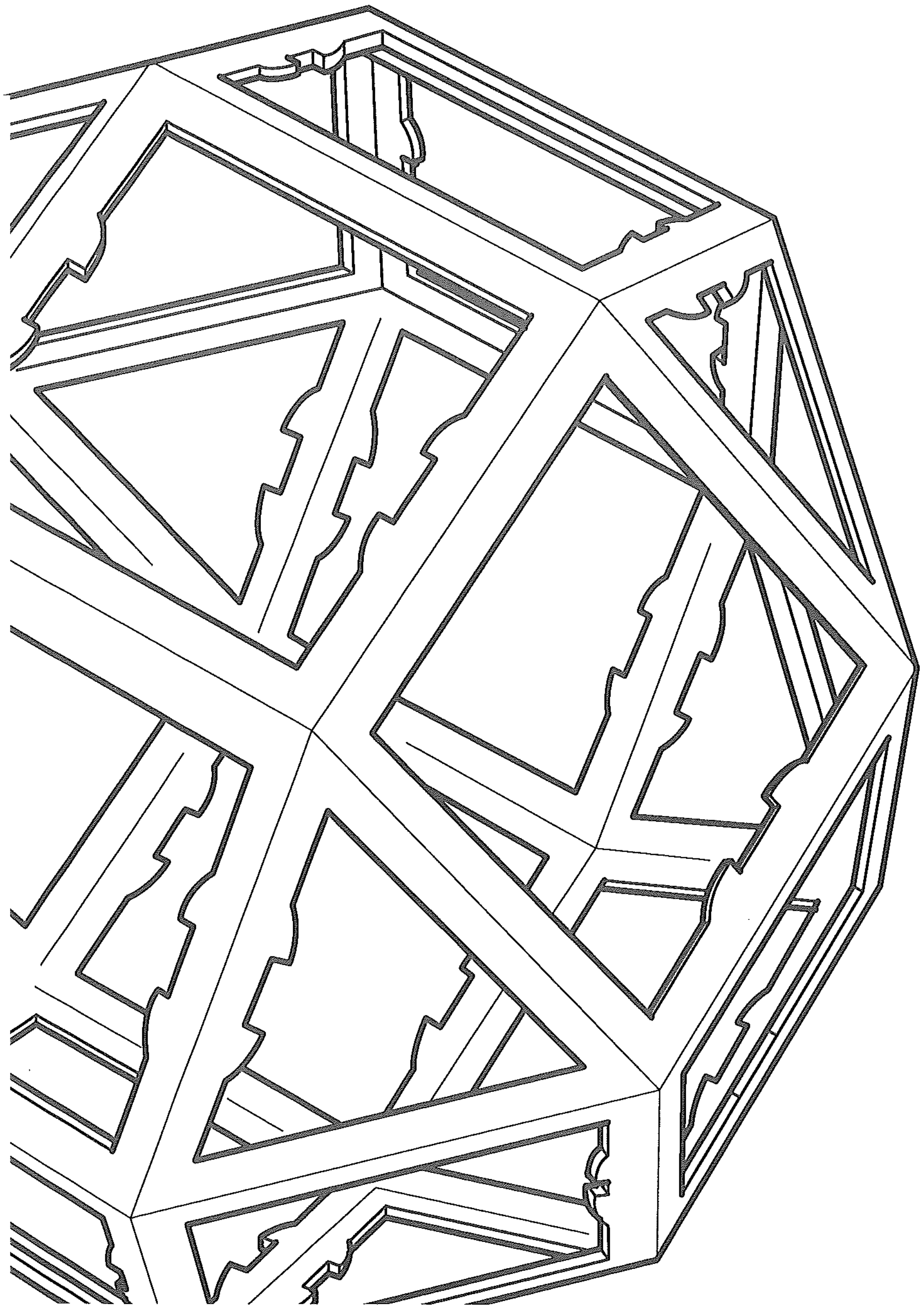
Summer Research Project

Topological Geometry

By Ash Lightfoot



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Topological Geometry

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Abstract

One may topologise an incidence structure by imposing that the operations of joining points and intersecting lines are continuous. This project investigates two different approaches to this. We first show that with respect to the canonical topology of the real affine and projective planes, these operations are continuous. Then, without a canonical topology at hand, we construct a topology on the line set of the Moulton planes so that these operations are continuous. Furthermore, we show that this topology is essentially unique.

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1 Introduction

Incidence structures in their own right form the well-studied field of incidence geometry. By describing the operations of joining points and intersecting lines as functions, we may form topologies on these structures by imposing continuity conditions on these functions. Topological geometry is the study of such topologies, and is relatively new.

The aim of this project is to investigate the construction and properties of topologies on two particular kinds of incidence structures: the real affine and projective planes, and the Moulton planes. The first part of the project gives a construction of the real projective plane and shows that the topology on the set of lines of both the real affine and projective planes that arises naturally from this construction implies continuity of the joining and intersection maps. The second part of the project presents a converse approach: we begin with the basic structure that is a Moulton plane and construct a topology on the set of lines based on the premise that the joining and intersection maps must be continuous with respect to this topology. The pinnacle of the project is reached when we show that this topology is essentially unique.

The underlying ideas behind many of the proofs presented are based on those of [2], but even in those cases we hope to have provided a service in fleshing out the details.

To appreciate this report, basic concepts of topology are assumed. However, we obey the conventions of [1] so that the reader can source the requisite topological definitions. In addition, on occasion we provide references to definitions as well as to proofs of well-known topological facts. Indeed, topological geometry provides a friendly application to the tools of topology and thus may aid the reader's own understanding. Of particular importance to us is the notion of a quotient space, and we provide proofs of some general theorems relating to this as the need arises.

1.1 Incidence Structures

We shall regard an incidence structure as we would an abstract space given to us with a certain structure, where our task is to extract an appropriate topological structure that satisfies some continuity conditions. It is therefore of less importance to us how the rules or theorems of these incidence structures come about; we shall consider them as given. We will, however, need to speak some of the language of incidence geometry in order to describe such rules, so let us outline our terminology.

An *incidence structure* is a triple $\mathcal{P} = (\mathbf{P}, \mathfrak{L}, \mathbf{F})$, where $\mathbf{P}$ is the set of *points*, $\mathfrak{L}$ is the set of *lines* and $\mathbf{F} \subseteq P \times \mathfrak{L}$ is called the set of *flags*. A point $\mathbf{p} \in P$ *lies on*, or is *incident with*, a line $L \in \mathfrak{L}$ if and only if $(p, L) \in \mathbf{F}$. We say that a set of points $Q \subseteq \mathbf{P}$ is collinear if there is a line $L \in \mathfrak{L}$ such that $\{(\mathbf{p}, L) : \mathbf{p} \in Q\}$ is contained in $\mathbf{F}$. For each point $\mathbf{p}$ we call the set of lines through $\mathbf{p}$ the *pencil* $\mathfrak{L}_p = \{L \in \mathfrak{L} : (\mathbf{p}, L) \in \mathbf{F}\}$ of $\mathbf{p}$. We may sometimes refer to the point set, line set and flag set of an incidence structure $\mathcal{P}$ by $\mathbf{P}(\mathcal{P})$, $\mathfrak{L}(\mathcal{P})$ and $\mathbf{F}(\mathcal{P})$, respectively.

We now give two concrete examples of incidence structures, which also serve to introduce the notions of joining points and intersecting lines.

An *affine plane* is an incidence structure $(\mathbf{P}, \mathfrak{L}, \mathbf{F})$ satisfying the following axioms:

(A1) Any two distinct points $\mathbf{p}$ and $\mathbf{q}$ are on a unique line, which we denote by $\mathbf{p} \vee \mathbf{q}$.

(A2) Given a line L and a point $\mathbf{p}$ such that $(\mathbf{p}, L) \notin \mathbf{P}$, there exists a unique line $M \in \mathfrak{L}_p$ such that L and M have no point in common.

(A3) There is a set of three points which are not collinear.

Note that axiom (A3) excludes the possibility of a single line on which all the points lie forming an affine plane.

A *projective plane* is an incidence structure satisfying the following axioms:

(P1) Any two distinct points $\mathbf{p}$ and $\mathbf{q}$ are on a unique line $\mathbf{p} \vee \mathbf{q}$.

(P2) Any two distinct lines L and M are incident with a unique point, called their *point of intersection* and denoted $L \wedge M$.

(P3) There is a set of four points, no three of which are collinear (we call such a set a *non-degenerate quadrangle*).

The third axiom in this case, in addition to ruling out the single-line situation as in the affine plane, excludes the possibility of the set of lines being three lines intersecting to form a triangle.

The geometric operations of joining points and intersecting lines are central to topological geometry. In the affine and projective planes we see from their axioms of definition that we can look to describe these operations as functions. The continuity of these functions provides the foundation for us to introduce topological notions.

2 The Real Affine and Projective Planes

After defining the real affine plane, we form its projective completion and shows that this is isomorphic to the real projective plane. Then we endow the line set of the real projective plane with a topology that arises naturally from its construction. Hence when show that, with respect to this topology, the joining and intersection maps on the real projective plane are continuous, we are able to infer the analogous result for the real affine plane.

2.1 The Real Affine Plane

The *real affine plane*, denoted $\mathbb{A}_2\mathbb{R}$, is the incidence structure $\mathbb{A}_2\mathbb{R} = (\mathbb{R}^2, \mathfrak{L}, \mathbf{F})$ where $\mathfrak{L}$ consists precisely of the following subsets of $\mathbb{R}^2$:

$$L_{[s,t]} = \{(x, sx+t) : x \in \mathbb{R}\} \text{ for } s,t \in \mathbb{R} \quad \text{(the tilted lines);}$$
$$L_{[c]} = \{(c, y) : y \in \mathbb{R}\} \text{ for } c \in \mathbb{R} \quad \text{(the vertical lines),}$$

and where $(\mathbf{p}, L) \in \mathbf{F}$ if and only if $\mathbf{p} \in L$. We say that the line $L_{[s,t]}$ has *slope* s and *intercept* t, and the line $L_{[c]}$ has slope ∞. Lines with the same slope are said to be *parallel*. Figure 2.1 illustrates the two tilted lines $L_{[1,1]}$ and $L_{[-\frac{1}{2},0]}$, and the vertical line $L_{[\frac{3}{2}]}$ in $\mathbb{A}_2\mathbb{R}$.

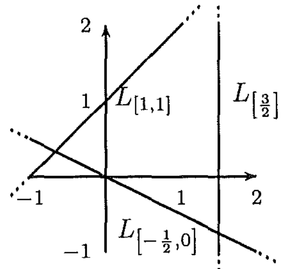


Figure 2.1

It is worthwhile to justify our terminology by verifying that the real affine plane satisfies the axioms (A1)-(A3) of an affine plane.

Theorem 2.1. *The real affine plane $\mathbb{A}_2\mathbb{R}$ is an affine plane.*

Proof. We show that the axioms (A1) to (A3) hold for $\mathbb{A}_2\mathbb{R}$.
(A1) Let $\mathbf{p} = (x_1, y_1), \mathbf{q} = (x_2, y_2)$ be two distinct points in $\mathbb{R}^2$. If $x_1 = x_2$, then $L_{[x_1]}$ is the unique line

joining $\mathbf{p}$ and $\mathbf{q}$. If $x_1 \neq x_2$, then $L_{[s,t]}$ joins $\mathbf{p}$ and $\mathbf{q}$ where $s = \frac{y_2 - y_1}{x_2 - x_1}$ and $t = y_1 - sx_1$, so $L_{[s,t]}$ is uniquely determined.
(A2) Let $\mathbf{p} = (x_1, y_1) \in \mathbb{R}^2$ and let L be a line not incident with $\mathbf{p}$. If $L = L_{[c]}$ for some $c \in \mathbb{R}$ then $M = L_{[x_1]}$ is the unique line through $\mathbf{p}$ parallel to L. If $L = L_{[s,t]}$ for some $s, t \in \mathbb{R}$, then take $M = L_{[s,t']}$, where $t' = y_1 - sx_1$.
(A3) The points $(0,0)$,$(1,0)$ and $(1,1)$ in $\mathbb{R}^2$ are not collinear. □

There are two additional properties of the real affine plane that will be useful.

Proposition 2.2. *In the real affine plane:*

(i) Two lines of different slope have a unique point of intersection.
(ii) Two different lines are disjoint (i.e., have no point of intersection) if and only if they have the same slope.

Proof.
(i) Let L, M be lines with distinct slopes $s, u \in \mathbb{R} \cup \{\infty\}$, respectively. If $s = \infty$, then $L = L_{[c]}$ for some $c \in \mathbb{R}$ and $M = L_{[u,v]}$ for some $v \in \mathbb{R}$. Setting $(c, y) = (x, ux + v)$, we obtain that the two lines intersect at the unique point $(c, uc + v)$. On the other hand, if $\{s, u\} \subseteq \mathbb{R}$ then $L = L_{[s,t]}$ and $M = L_{[u,v]}$ for some $t, v \in \mathbb{R}$; these two lines intersect at the unique point $(-\frac{t-v}{s-u}, -s\frac{t-v}{s-u} + t)$.
(ii) By part (i), two lines of different slope intersect. Conversely, suppose that L and M are different lines of the same slope that intersect at a point $\mathbf{p} = (x, y)$. If their slope is ∞, then we have that $L = M = L_{[x]}$ — a contradiction — so their slope is s for some $s \in \mathbb{R}$. But then $L = M = L_{[s,t]}$ where $t = y - sx$ — again, a contradiction. Hence L and M must not intersect. □

2.2 Projective Completion

With the aim of finding a correspondence between the real affine and projective planes, we construct the *projective completion* of the real affine plane, denoted $\overline{\mathbb{A}_2\mathbb{R}}$. Beginning with $\mathbb{A}_2\mathbb{R}$, for each parallel class we add a point to the point-set called a *point at infinity* so that all the lines in the parallel class intersect at this point. We then add a line, called the *line at infinity*, to the line-set which passes through precisely these new points. For each slope $s \in \mathbb{R} \cup \{\infty\}$, we denote by (s) the point at infinity corresponding to lines of that slope; the line at infinity is denoted $L_{[\infty]} = \{(s) : s \in \mathbb{R} \cup \{\infty\}\}$. Thus $\overline{\mathbb{A}_2\mathbb{R}}$ is the incidence structure $\overline{\mathbb{A}_2\mathbb{R}} = (\overline{P}, \overline{\mathfrak{L}}, \overline{\mathbf{F}})$, where $\overline{P} = \mathbb{R}^2 \cup \{(s) : s \in \mathbb{R} \cup \{\infty\}\}$, $\overline{\mathfrak{L}} = \mathfrak{L}(\mathbb{A}_2\mathbb{R}) \cup \{L_{[\infty]}\}$ and $\overline{\mathbf{F}} = \mathbf{F}(\mathbb{A}_2\mathbb{R}) \cup \{((s), L_{[\infty]}) : s \in \mathbb{R} \cup \{\infty\}\}$. Figure 2.2 illustrates the effect of the projective completion on the lines of Figure 2.1 in the construction of $\overline{\mathbb{A}_2\mathbb{R}}$.

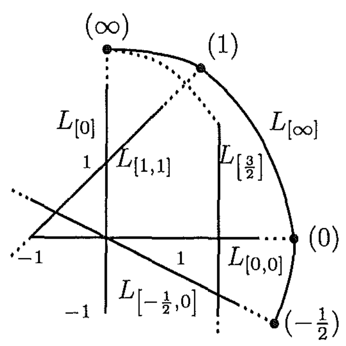


Figure 2.2

Let us justify terminology once more and show that projective completion does indeed produce a projective plane.

Theorem 2.3. *The projective completion of the real affine plane is a projective plane.*

Proof. We verify the axioms (P1)-(P3) are satisfied by $\overline{\mathbb{A}_2\mathbb{R}}$.
(P1) Let $\mathbf{p}$ and $\mathbf{q}$ be distinct points in $\mathbb{R}^2 \cup \{(s) : s \in \mathbb{R} \cup \{\infty\}\}$. If $\{\mathbf{p}, \mathbf{q}\} \subseteq \mathbb{R}^2$ then the unique line $\mathbf{p} \vee \mathbf{q}$ in $\mathfrak{L}(\mathbb{A}_2\mathbb{R})$ is the unique line in $\overline{\mathfrak{L}}$ joining $\mathbf{p}$ and $\mathbf{q}$ since $L_{[\infty]}$ does not contain points in $\mathbb{R}^2$. Now suppose that $\mathbf{p} = (s)$ for some $s \in \mathbb{R} \cup \{\infty\}$, and $\mathbf{q} \in \mathbb{R}^2$. By (A2) for $\mathbb{A}_2\mathbb{R}$, given that $\mathbf{q}$ is not incident with $L_{[s]}$, there is a unique line $M \in \mathfrak{L}(\mathbb{A}_2\mathbb{R})$ with slope s through $\mathbf{q}$; this line is also incident with (s). Finally, if $\{\mathbf{p}, \mathbf{q}\} \subseteq \{(s) : s \in \mathbb{R} \cup \{\infty\}\}$, then $L_{[\infty]}$ is the unique line joining $\mathbf{p}$ and $\mathbf{q}$.
(P2) Let L and M be distinct lines in $\overline{\mathfrak{L}}$. Suppose that $\{L, M\} \subseteq \mathfrak{L}$. If L and M are not parallel then they intersect at a unique point by Proposition 2.2(i); if L and M are parallel then they intersect precisely at a point at infinity. In the case where exactly one of the lines is $L_{[\infty]}$, they intersect uniquely at a point at infinity.
(P3) The points $(0,0), (0,1), (1,0)$ and $(1,1)$ form a non-degenerate quadrangle. □

2.3 The Real Projective Plane

The *real projective plane*, denoted $\mathbb{P}_2\mathbb{R}$, is the incidence structure $\mathbb{P}_2\mathbb{R} = (P_2\mathbb{R}, \mathcal{L}_2\mathbb{R}, \mathbf{F}_2\mathbb{R})$, where $P_2\mathbb{R}$, $\mathcal{L}_2\mathbb{R}$ and $\mathbf{F}_2\mathbb{R}$ are defined as follows. Considering $\mathbb{R}^3$ as a vector space over $\mathbb{R}$, the points of $P_2\mathbb{R}$ are precisely the 1-dimensional subspaces of $\mathbb{R}^3$; the lines of $\mathcal{L}_2\mathbb{R}$ are the 2-dimensional subspaces. Since two distinct 2-dimensional subspaces in $\mathbb{R}^3$ intersect at a unique 1-dimensional subspace (cf. axiom (P2)), and conversely two distinct 1-dimensional subspaces uniquely define a 2-dimensional subspace in $\mathbb{R}^3$ (cf. axiom (P1)), it is clear that $\mathbb{P}_2\mathbb{R}$ is indeed a projective plane. We adopt the following notation for the points and lines. The point set is

$$P_2\mathbb{R} = \{[(x_1, x_2, x_3)] : (x_1, x_2, x_3) \in \mathbb{R}^3 \backslash \{(0,0,0)\}\},$$

where $[\mathbf{v}]$ denotes the linear span of $\mathbf{v} \in \mathbb{R}^3$. Since the 2-dimensional subspaces of $\mathbb{R}^3$ correspond bijectively with the orthogonal complements (with respect to the Euclidean dot product) of 1-dimensional subspaces, the line set is

$$\mathcal{L}_2\mathbb{R} = \{[(x_1, x_2, x_3)]^{\perp} : (x_1, x_2, x_3) \in \mathbb{R}^3 \backslash \{(0,0,0)\}\}.$$

2.4 Correspondence between $\mathbb{P}_2\mathbb{R}$ and $\overline{\mathbb{A}_2\mathbb{R}}$

Later we will show that the real projective plane has a natural topology; to use this topology in the context of the real affine plane we need a relationship between $\mathbb{A}_2\mathbb{R}$ and $\mathbb{P}_2\mathbb{R}$. If $\mathcal{P}_1 = (\mathbf{P}_1, \mathfrak{L}_1, \mathbf{F}_1)$ and $\mathcal{P}_2 = (\mathbf{P}_2, \mathfrak{L}_2, \mathbf{F}_2)$ are both affine (or both projective) planes then a map $\gamma : \mathbf{P}_1 \to \mathbf{P}_2$ between their point-sets is called a *isomorphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if γ is bijective and

$$\gamma(\mathbf{p} \vee \mathbf{q}) = \gamma(\mathbf{p}) \vee \gamma(\mathbf{q})$$

for every pair of distinct points $\mathbf{p}, \mathbf{q}$ in $\mathbf{P}_1$. That is, γ maps collinear points to collinear points. If there is an isomorphism between two affine (or projective) planes we say that the two planes are *isomorphic* and write $\mathcal{P}_1 \cong \mathcal{P}_2$.

We proceed to define an isomorphism from $\overline{\mathbb{A}_2\mathbb{R}}$ to $\mathbb{P}_2\mathbb{R}$. Denote

$$W := \mathbb{R}^2 \times \{0\} = [(0,0,1)]^{\perp} \in \mathcal{L}_2\mathbb{R}$$

and define the incidence structure $\mathbb{P}_2\mathbb{R}\backslash W = (P_2\mathbb{R}\backslash W, \mathcal{L}_2\mathbb{R}\backslash W, \mathbf{F}\backslash W)$ to be that obtained from $\mathbb{P}_2\mathbb{R}$ by removing the line W from $\mathcal{L}_2\mathbb{R}$ and removing all the points incident with W (*i.e.*, the 1-dimensional subspaces contained in $\mathbb{R}^2 \times \{0\}$) from $P_2\mathbb{R}$. Thus,

$$P_2\mathbb{R}\backslash W = \{[(x_1, x_2, 1)] : (x_1, x_2) \in \mathbb{R}^2\}$$

and

$$\mathcal{L}_2\mathbb{R}\backslash W = \left\{[(x_1, x_2, x_3)]^\perp : (x_1, x_2) \in \mathbb{R}^2\backslash\{(0,0)\}\right\}.$$

Theorem 2.4. *The map $\psi : \mathbb{R}^2 \to \mathbb{P}_2\mathbb{R}\backslash W$, defined by*

$$\psi(x, y) = [(x, y, 1)],$$

is an isomorphism from $\mathbb{A}_2\mathbb{R}$ to $\mathbb{P}_2\mathbb{R}\backslash W$. In particular, ψ maps the lines

$$L_{[s,t]} \mapsto [(-s, 1, -t)]^\perp;$$
$$L_{[c]} \mapsto [(1, 0, -c)]^\perp.$$

Proof. The map $\mathbb{P}_2\mathbb{R}\backslash W \to \mathbb{R}^2$ given by $[(x, y, z)] \mapsto (\frac{x}{z}, \frac{y}{z})$ is the inverse mapping of ψ, showing that ψ is bijective. We are left to verify that ψ maps each line in $\mathfrak{L}(\mathbb{A}_2\mathbb{R})$ to a line in $\mathcal{L}_2\mathbb{R}\backslash W$. Consider $L_{[s,t]} \in \mathfrak{L}(\mathbb{A}_2\mathbb{R})$. For each $(x, sx + t) \in L_{[s,t]}$, since

$$(x, sx + t, 1) \cdot (-s, 1, -t) = -sx + sx + t - t = 0,$$

we have that

$$\psi(x, sx + t) = [(x, sx + t, 1)] \in [(-s, 1, -t)]^\perp$$

and hence $\psi(L_{[s,t]}) \subseteq [(-s, 1, -t)]^\perp$. Conversely, let $[(x_1, x_2, 1)] \in [(-s, 1, -t)]^\perp$. Then $-sx_1 + x_2 - t = 0$, so

$$[(x_1, x_2, 1)] = [(x_1, sx_1 + t, 1)] \in \psi(L_{[s,t]}).$$

Thus $\psi(L_{[s,t]}) = [(-s, 1, -t)]^\perp \in \mathcal{L}_2\mathbb{R}\backslash W$.

Now consider $L_{[c]} \in \mathfrak{L}(\mathbb{A}_2\mathbb{R})$. For each $(c, y) \in L_{[c]}$, since

$$(c, y, 1) \cdot (1, 0, -c) = c - c = 0,$$

we have $\psi(c, y) = [(c, y, 1)] \in [(1, 0, -c)]^\perp$ and so $\psi(L_{[c]}) \subseteq [(1, 0, -c)]^\perp$. Conversely, let $[(x_1, x_2, 1)] \in [(1, 0, -c)]^\perp$. Then $x_1 - c = 0$ and so

$$[(x_1, x_2, 1)] = [(c, x_2, 1)] \in \psi(L_{[c]}).$$

Thus $\psi(L_{[c]}) = [(1, 0, -c)]^\perp \in \mathcal{L}_2\mathbb{R}\backslash W$. □

With the aim of extending ψ to an isomorphism from $\overline{\mathbb{A}_2\mathbb{R}}$ to $\mathbb{P}_2\mathbb{R}$, we first establish that the projective completion of $\mathbb{P}_2\mathbb{R}\backslash W$, denoted $\overline{\mathbb{P}_2\mathbb{R}\backslash W}$, is isomorphic to $\mathbb{P}_2\mathbb{R}$.

Proposition 2.5. *The projective completion of $\mathbb{P}_2\mathbb{R}\backslash W$ is isomorphic to the real projective plane.*

Proof. We are required to show that the parallel classes of $\mathbb{P}_2\mathbb{R}\backslash W$, as given in Theorem 2.4, intersect precisely at the points of W. Let L_1 and L_2 be two distinct, parallel lines in $\mathcal{L}_2\mathbb{R}\backslash W$. Consider first the case where L_1 and L_2 are of the form $[(-s, 1, -t)]^\perp$ and $L_2 = [(-s, 1, -v)]^\perp$, respectively. Suppose that $[(x, y, z)] \in L_1 \wedge L_2$; then we have

$$-xs + y - tz = 0 \text{ and}$$
$$-xs + y - vz = 0,$$

so $z(t - v) = 0$. Since $t \neq v$, this implies that $z = 0$; hence $[(x, y, z)] = [(x, y, 0)] \in W$. On the other hand, suppose that L_1 and L_2 are of the form $[(1, 0, -c)]^\perp$ and $[(1, 0, -d)]^\perp$. Then, for $[(x, y, z)] \in L_1 \wedge L_2$, we have

$$x - cz = 0 \text{ and}$$
$$x - dz = 0,$$

so $z(c-d)=0$ and hence $z=0$. Thus $[(x,y,z)]=[(x,y,0)]\in W$, so $L_1\wedge L_2\subseteq W$.

Conversely, points in W are of the form $[(x,y,0)]$ with $(x,y)\in\mathbb{R}^2\backslash\{(0,0)\}$ and thus can be written either as $\left[(1,\frac{y}{x},0)\right]^\perp\in\bigcap_{t\in\mathbb{R}}\left[(-\frac{y}{x},1,-t)\right]^\perp$ or as $[(0,y,0)]^\perp\in\bigcap_{c\in\mathbb{R}}[(1,0,-c)]^\perp$. □

Theorem 2.6. *The map $\overline{\psi}:\overline{\mathbb{A}_2\mathbb{R}}\to\mathbb{P}_2\mathbb{R}$ defined by*

$$(x,y)\mapsto[(x,y,1)],\ (s)\mapsto[(1,s,0)],\ (\infty)\mapsto[(0,1,0)],$$

is an isomorphism from $\overline{\mathbb{A}_2\mathbb{R}}$ to $\mathbb{P}_2\mathbb{R}$. In particular, $\overline{\psi}$ maps the lines

$$L_{[s,t]}\mapsto[(-s,1,-t)]^\perp;\ L_{[c]}\mapsto[(1,0,-c)]^\perp;\ L_{[\infty]}\mapsto[(0,0,1)]^\perp.$$

Proof. The mapping $\mathbb{P}_2\mathbb{R}\to\overline{\mathbb{A}_2\mathbb{R}}$ given by

$$[(x,y,0)]\mapsto(\tfrac{y}{x})\text{ for }x\neq 0;\ [(0,y,0)]\mapsto(\infty);\ [(x,y,1)]\mapsto(x,y),$$

is the inverse of $\overline{\psi}$, so $\overline{\psi}$ is a bijection. We now verify that $\overline{\psi}$ takes lines in $\overline{\mathbb{A}_2\mathbb{R}}$ to lines in $\mathbb{P}_2\mathbb{R}$, as prescribed in the theorem statement. Since

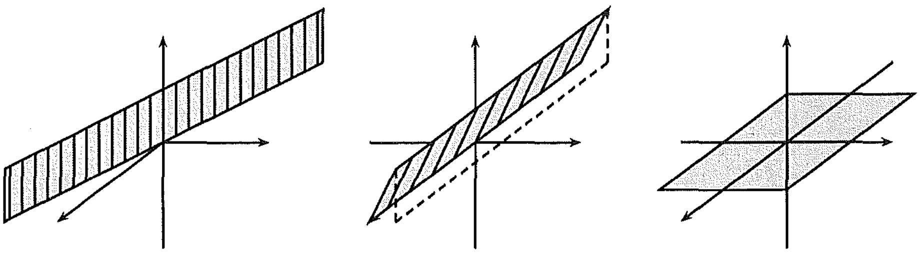
$$\begin{aligned}(x,y)\in L_{[s,t]}&\Leftrightarrow -sx+y-t=0\\ &\Leftrightarrow[(x,y,1)]\in[(-s,1,-t)]^\perp\end{aligned}$$

and

$$\begin{aligned}(x,y)\in L_{[c]}&\Leftrightarrow x-c=0\\ &\Leftrightarrow[(x,y,1)]\in[(1,0,-c)]^\perp,\end{aligned}$$

we see that $\overline{\psi}(L_{[s,t]})=[(-s,1,-t)]^\perp$ and $\overline{\psi}(L_{[c]})=[(1,0,-c)]^\perp$, respectively.

On the other hand, from the definition of $\overline{\psi}$ we see that the points on $L_{[\infty]}$ are mapped into $[(0,0,1)]^\perp$. Conversely, let $[(x,y,0)]\in[(0,0,1)]^\perp$. For $x\neq 0$, we have $[(x,y,0)]=\overline{\psi}(\frac{y}{x})$; for $x=0$, we have $[(0,y,0)]=\overline{\psi}(\infty)$. Hence $[(0,0,1)]^\perp\subseteq\overline{\psi}(L_{[\infty]})$ and we deduce that $\overline{\psi}(L_{[\infty]})=[(0,0,1)]^\perp$. □



(a) Tilted Lines (b) Vertical Lines (c) Line at Infinity

Figure 2.3: Images of lines in $\mathbb{A}_2\mathbb{R}$ under $\overline{\psi}$ of Theorem 2.6

2.5 The Topology of $\mathbb{P}_2\mathbb{R}$

Endow $\mathbb{R}^3$ with the Euclidean topology and denote $\mathbf{0}:=(0,0,0)$. From their definitions, it is natural to define the topologies on $P_2\mathbb{R}$ and $\mathcal{L}_2\mathbb{R}$ as the quotient spaces formed by identifying points of $\mathbb{R}^3\backslash\{\mathbf{0}\}$ which determine the same 1-dimensional or 2-dimensional subspace, respectively. That is,

$$P_2\mathbb{R}=(\mathbb{R}^3\backslash\{\mathbf{0}\})/\sim\quad\text{where }\mathbf{x}\sim\mathbf{y}\Leftrightarrow[\mathbf{x}]=[\mathbf{y}]$$

and

$$\mathcal{L}_2\mathbb{R} = (\mathbb{R}^3\backslash\{\mathbf{0}\})/\sim \quad \text{where } \mathbf{x} \sim \mathbf{y} \Leftrightarrow [\mathbf{x}]^\perp = [\mathbf{y}]^\perp.$$

More precisely, the topologies of $P_2\mathbb{R}$ and $\mathcal{L}_2\mathbb{R}$ are defined by the quotient maps (cf. [1], Chapter 2.11)

$$\begin{aligned} \pi &: \mathbb{R}^3\backslash\{\mathbf{0}\} \to P_2\mathbb{R} : \ \mathbf{x} \mapsto [\mathbf{x}] \\ \lambda &: \mathbb{R}^3\backslash\{\mathbf{0}\} \to \mathcal{L}_2\mathbb{R} : \ \mathbf{x} \mapsto [\mathbf{x}]^\perp. \end{aligned}$$

We endow $P_2\mathbb{R} \times \mathcal{L}_2\mathbb{R}$ with the product topology (cf. [1], Chapter 2.8) and consider the flag set $\mathbf{F} \subseteq P_2\mathbb{R} \times \mathcal{L}_2\mathbb{R}$ as a subspace called the *flag space*. We also endow the sets of distinct points and distinct lines,

$$(P_2\mathbb{R})^2_* := (P_2\mathbb{R})^2\backslash\{(\mathbf{p},\mathbf{p}) : \mathbf{p} \in P_2\mathbb{R}\}$$

and

$$(\mathcal{L}_2\mathbb{R})^2_* := (\mathcal{L}_2\mathbb{R})^2\backslash\{(L,L) : L \in \mathcal{L}_2\mathbb{R}\}$$

respectively, with the respective subspace topologies. Our aim is to show that with respect to these topologies, the joining and intersection maps are continuous.

We begin by expressing the inherent duality in the definitions of the point and line sets by a homeomorphism.

Lemma 2.7. *The quotient maps π and λ are open maps.*

Proof. We consider the map π only; proof for the map λ is completely analogous. Let U be an open subset of $\mathbb{R}^3\backslash\{\mathbf{0}\}$. By definition, $\pi(U) = \{[\mathbf{x}] : \mathbf{x} \in U\}$ is open in $P_2\mathbb{R}$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^3\backslash\{\mathbf{0}\}$. Now,

$$\begin{aligned} \pi^{-1}(\pi(U)) &= \{\mathbf{y} : [\mathbf{y}] = [\mathbf{x}],\ \mathbf{x} \in U\} \\ &= \{k\mathbf{x} : \mathbf{x} \in U,\ k \in \mathbb{R}\backslash\{0\}\}, \end{aligned}$$

so it suffices to show that for each $k \in \mathbb{R}\backslash\{0\}$, the set $kU := \{k\mathbf{x} : \mathbf{x} \in U\}$ is open in $\mathbb{R}^3\backslash\{\mathbf{0}\}$.

Let $k\mathbf{x} \in kU$; since U is open in $\mathbb{R}^3\backslash\{\mathbf{0}\}$ there is a $\delta > 0$ such that if $\mathbf{y} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$ with $||\mathbf{x} - \mathbf{y}|| < \delta$, then $\mathbf{y} \in U$. Thus, for each $\mathbf{y}' \in \mathbb{R}^3\backslash\{\mathbf{0}\}$, letting $\mathbf{y}' = k\mathbf{y}$, we have that

$$||\mathbf{x} - \mathbf{y}|| < \delta \Leftrightarrow ||k\mathbf{x} - \mathbf{y}'|| < k\delta$$

implies that $\mathbf{y}' \in kU$. Hence kU is open in $\mathbb{R}^3\backslash\{\mathbf{0}\}$, and we deduce that π is an open map. □

Theorem 2.8. *The map $\varphi := \lambda \circ \pi^{-1} : [\mathbf{x}] \to [\mathbf{x}]^\perp$ is a homeomorphism from $P_2\mathbb{R}$ to $\mathcal{L}_2\mathbb{R}$. Furthermore, the joining and intersection maps*

$$\begin{aligned} \vee &: (P_2\mathbb{R})^2_* \to \mathcal{L}_2\mathbb{R} : (\mathbf{p},\mathbf{q}) \mapsto \mathbf{p} \vee \mathbf{q}, \ \textit{and} \\ \wedge &: (\mathcal{L}_2\mathbb{R})^2_* \to P_2\mathbb{R} : (L, M) \mapsto L \wedge M \end{aligned}$$

satisfy $\varphi \circ \wedge = \vee \circ \varphi^{-1}$.

Proof. Since the maps π and λ are continuous and open (Lemma 2.7), we see that the evidently-bijective map φ is a homeomorphism. Now, since $\vee([\mathbf{x}],[\mathbf{y}]) = [\mathbf{z}]^\perp$ if and only if $\mathbf{x} \cdot \mathbf{z} = \mathbf{0} = \mathbf{y} \cdot \mathbf{z}$, which in turn holds if and only if $\wedge([\mathbf{x}]^\perp, [\mathbf{y}]^\perp) = [\mathbf{z}]$, we have that

$$\vee([\mathbf{x}],[\mathbf{y}]) = [\mathbf{z}]^\perp = \varphi([\mathbf{z}]) = \varphi(\wedge([\mathbf{x}]^\perp,[\mathbf{y}]^\perp)) = \varphi(\wedge(\varphi([\mathbf{x}]),\varphi([\mathbf{y}])).$$

Thus $\vee = \varphi \circ \wedge \circ \varphi$, and so $\varphi \circ \wedge = \vee \circ \varphi^{-1}$. □

Let X^∞ denote the set of all the sequences in a space X. The abbreviation $(x_n) := \{x_n \in X : n \in \mathbb{N}\}$ is used to denote such a sequence in X.

Lemma 2.9. *Let X, Y be Hausdorff spaces, with Y be compact and let $f : X \to Y$ be a map. Suppose that the* graph of f,

$$\mathrm{Graph}(f) := \{(x, f(x)) : x \in X\},$$

is closed in $X \times Y$ (endowed with the product topology). Then f is continuous.

Proof. Let $(x_n) \in X^\infty$ and suppose that (x_n) converges to $x \in X$. We wish to show that $(f(x_n)) \xrightarrow[n\to\infty]{} f(x)$. Since Y is compact, $(f(x_n)) \in Y^\infty$ has a subsequence $(f(x_{n_k}))$ converging to a unique point $y \in Y$. Thus the sequence $(x_{n_k}, f(x_{n_k})) \in (X \times Y)^\infty$ converges to (x, y), which, as Graph(f) is closed, implies that $(x, y) \in \mathrm{Graph}(f)$; whence $y = f(x)$, and f is continuous. $\square$

In order to apply Lemma 2.9 to the joining map $\vee : (P_2\mathbb{R})^2_* \to \mathcal{L}_2\mathbb{R}$, we first establish that the spaces $P_2\mathbb{R}$ and $\mathcal{L}_2\mathbb{R}$ are Hausdorff and compact. We require some general results concerning quotient spaces.

Lemma 2.10. *Let X, Y be spaces such that $p : X \to Y$ is a quotient map, and let Z be a space. Then a map $f : Y \to Z$ is continuous if and only if the composite map $f \circ p : X \to Z$ is continuous.*

Proof. As p is continuous, if f is continuous then so is the composition $f \circ p$. Conversely, suppose that $f \circ p$ is continuous. Let U be an open set in Z. By the continuity of $f \circ p$, we have that

$$(f \circ p)^{-1}(U) = p^{-1}(f^{-1}(U))$$

is open in X, and hence $f^{-1}(U)$ is open in Y since p is a quotient map. $\square$

Theorem 2.11. *Let X, Y, X^* and Y^* be spaces and let $p : X \to X^*$, $q : Y \to Y^*$ be quotient maps. Let $g : X \to Y$ and $h : X^* \to Y^*$ be maps such that g is a homeomorphism and h is a bijection satisfying the commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ X^* \arrow[r, "h"'] & Y^* \end{tikzcd}$$

Then h is a homoeomorphism.

Proof. Since $h \circ p = q \circ g$ is continuous, by Lemma 2.10 the map h is continuous. Now let U be an open subset of X^*. Then $p^{-1}(U)$ is open in X and hence $q^{-1}(h(U)) = g(p^{-1}(U))$ is open in Y since g is an open map. Thus $h(U)$ is open in Y^*, so h is an open map. Whence h is a homeomorphism. $\square$

To prove that $P_2\mathbb{R}$ is Hausdorff we will show that each pair of distinct points lies in a subspace homeomorphic to the Hausdorff space $\mathbb{R}^2$. Let $\mathbf{e}_i$, $i = 1, 2, 3$, denote the standard unit vectors of $\mathbb{R}^3$ and recall that $W = [\mathbf{e}_3]^\perp$; denote the standard basis $E := \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. For $\mathbf{y} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$, denote the set

$$P_2\mathbb{R}\backslash[\mathbf{y}]^\perp := \{[\mathbf{x}] : \mathbf{x} \in (\mathbb{R}^3\backslash\{\mathbf{0}\})\backslash[\mathbf{y}]^\perp\}.$$

Lemma 2.12. *For each $\mathbf{h} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$, there is a homeomorphism $[A] : P_2\mathbb{R} \to P_2\mathbb{R}$ mapping $P_2\mathbb{R}\backslash W \mapsto P_2\mathbb{R}\backslash[\mathbf{h}]^\perp$.*

Proof. Given $\mathbf{h} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$, choose points $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathbb{R}^3$ so that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{h}\}$ is an orthogonal basis for $\mathbb{R}^3$. Let A be the matrix representation, with respect to E, of the linear map which takes E to B; that is, $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{h}]$. Then the inverse of A is the matrix representation, with respect to E, of the linear map taking B to E; whence[1] $A \in GL_3\mathbb{R}$.

[1]Denote the set of 3×3 invertible matrices over $\mathbb{R}$ by $GL_3\mathbb{R}$.

For $C = (C_{ij}) \in GL_3\mathbb{R}$, the map $\mathbb{R}^3\backslash\{0\} \to \mathbb{R}^3\backslash\{0\} : \mathbf{x} \mapsto C\mathbf{x}$ is continuous since the coordinate maps $x_i \mapsto \sum_{j=1}^{3} C_{ij}x_j$ are continuous; hence $\mathbf{x} \mapsto A\mathbf{x}$ is a homeomorphism. Now define the map $[A] : P_2\mathbb{R} \to P_2\mathbb{R}$ by $[\mathbf{x}] \mapsto [A\mathbf{x}]$; then $[A]^{-1} = [A^{-1}]$, so $[A]$ is a bijection. We thus obtain the commutative diagram

$$\begin{tikzcd} \mathbb{R}^3\backslash\{0\} \arrow[r, "A"] \arrow[d, "\pi"'] & \mathbb{R}^3\backslash\{0\} \arrow[d, "\pi"] \\ P_2\mathbb{R} \arrow[r, "{[A]}"'] & P_2\mathbb{R} \end{tikzcd}$$

satisfying the hypotheses of Theorem 2.11, so $[A]$ is a homeomorphism. That is, $[A]$ is a homeomorphism $P_2\mathbb{R} \to P_2\mathbb{R}$ such that $[A] : W \to [\mathbf{h}]^\perp$. Thus,

$$\begin{aligned} [A](P_2\mathbb{R}\backslash W) &= [A](\{[\mathbf{x}] : \mathbf{x} \in (\mathbb{R}^3\backslash\{0\})\backslash[\mathbf{e}_3]^\perp\}) \\ &= \{[A\mathbf{x}] : \mathbf{x} \in (\mathbb{R}^3\backslash\{0\})\backslash[\mathbf{e}_3]^\perp\} \\ &= \{[\mathbf{x}'] : \mathbf{x}' \in (\mathbb{R}^3\backslash\{0\})\backslash[\mathbf{h}]^\perp\} \\ &= P_2\mathbb{R}\backslash[\mathbf{h}]^\perp. \end{aligned}$$

□

Lemma 2.13. *The set*

$$P_2\mathbb{R}\backslash W := \{[\mathbf{x}] : \mathbf{x} \in (\mathbb{R}^3\backslash\{0\})\backslash[\mathbf{e}_3]^\perp\}$$

is open in $P_2\mathbb{R}$ and homeomorphic to $\mathbb{R}^2$.

Proof. We first show that $P_2\mathbb{R}\backslash W$ is open in $P_2\mathbb{R}$. Since $W = \mathbb{R}^2 \times \{0\}$ is closed in $\mathbb{R}^3$, its complement $\mathbb{R}^3\backslash W \subseteq \mathbb{R}^3\backslash\{0\}$ is open in $\mathbb{R}^3\backslash\{0\}$. Thus,

$$\pi^{-1}(P_2\mathbb{R}\backslash W) = \{\mathbf{x} \in \mathbb{R}^3\backslash\{0\} : \mathbf{x} \notin W\} = \mathbb{R}^3\backslash W$$

is open in $\mathbb{R}^3\backslash\{0\}$, so $P_2\mathbb{R}\backslash W$ is open in $P_2\mathbb{R}$ since π is a quotient map.

We now show that the map $\phi : \mathbb{R}^2 \to P_2\mathbb{R}\backslash W : (x_1, x_2) \mapsto [(x_1, x_2, 1)]$ is a homeomorphism. The map $\phi^{-1} : \mathbb{R}^2 \to P_2\mathbb{R}\backslash W$ given by $[(y_1, y_2, y_3)] \mapsto (\frac{y_1}{y_3}, \frac{y_2}{y_3})$ is well-defined since

$$\phi^{-1}([k(y_1, y_2, y_3)]) = \left(\frac{ky_1}{ky_3}, \frac{ky_2}{ky_3}\right) = \left(\frac{y_1}{y_3}, \frac{y_2}{y_3}\right)$$

for all $k \in \mathbb{R}\backslash\{0\}$. Furthermore, for $[(y_1, y_2, y_3)] \in P_2\mathbb{R}\backslash W$, we have that $y_3 \neq 0$ and thus

$$(\phi \circ \phi^{-1})([(y_1, y_2, y_3)]) = \phi\left(\left(\frac{y_1}{y_3}, \frac{y_2}{y_3}\right)\right) = \left[\left(\frac{y_1}{y_3}, \frac{y_2}{y_3}, 1\right)\right] = [(y_1, y_2, y_3)].$$

On the other hand, for $(x_1, x_2) \in \mathbb{R}^2$,

$$(\phi^{-1} \circ \phi)\big((x_1, x_2)\big) = \phi^{-1}([(x_1, x_2, 1)]) = (x_1, x_2).$$

Hence ϕ has inverse ϕ^{-1} and so is a bijection.

The map ϕ is continuous since we may write it as the composition of the continuous maps $\phi = \pi \circ f$:

$$\phi : (x_1, x_2) \overset{f}{\longmapsto} (x_1, x_2, 1) \overset{\pi}{\longmapsto} [(x_1, x_2, 1)].$$

On the other hand, we may write the composite map $\phi^{-1} \circ \pi : (y_1, y_2, y_3) \mapsto (\frac{y_1}{y_3}, \frac{y_2}{y_3})$ as the composition of the continuous maps g and h given by

$$(y_1, y_2, \underbrace{y_3}_{\neq 0}) \overset{h}{\longmapsto} \frac{1}{y_3}(y_1, y_2, 1) \overset{g}{\longmapsto} \frac{1}{y_3}(y_1, y_2),$$

so by Lemma 2.10 we obtain that ϕ^{-1} is continuous. Thus ϕ is a homeomorphism. □

Proposition 2.14. *Let* $\mathbf{h} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$. *Then the set* $P_2\mathbb{R} \backslash [\mathbf{h}]^\perp$ *is open in* $P_2\mathbb{R}$ *and homeomorphic to* $\mathbb{R}^2$.

Proof. Combine Lemmas 2.12 and 2.13. □

We are now in a position to prove that $P_2\mathbb{R}$ and $\mathcal{L}_2\mathbb{R}$ have the properties we desire.

Proposition 2.15. *The spaces* $P_2\mathbb{R}$ *and* $\mathcal{L}_2\mathbb{R}$ *are Hausdorff and compact.*

Proof. By Theorem 2.8 we need only consider $P_2\mathbb{R}$. Let $[\mathbf{x}]$ and $[\mathbf{y}]$ be distinct points in $P_2\mathbb{R}$. Then $\{[\mathbf{x}], [\mathbf{y}]\} \subseteq P_2\mathbb{R} \backslash [\mathbf{h}]^\perp$ where $\mathbf{h} = \mathbf{x} \boxtimes \mathbf{y}$ (with $\boxtimes$ denoting the vector cross-product in $\mathbb{R}^3$). By Proposition 2.14, $P_2\mathbb{R} \backslash [\mathbf{h}]^\perp$ is Hausdorff so there are disjoint open subsets U, V in $P_2\mathbb{R} \backslash [\mathbf{h}]^\perp$ containing $\mathbf{x}$ and $\mathbf{y}$, respectively. Furthermore, by Proposition 2.14, $P_2\mathbb{R} \backslash [\mathbf{h}]^\perp$ is open in $P_2\mathbb{R}$; hence U and V are open in $P_2\mathbb{R}$ as well. This shows that $P_2\mathbb{R}$ is Hausdorff.

Finally, we show that $P_2\mathbb{R}$ is the continuous image of the compact space[2]

$$\mathbb{S}^2 := \{\mathbf{x} \in \mathbb{R}^3 : ||\mathbf{x}|| = 1\}$$

and hence is compact ([1], Theorem 3.5.5). For each $[\mathbf{x}] \in P_2\mathbb{R}$, we have

$$[\mathbf{x}] = \left[\tfrac{\mathbf{x}}{||\mathbf{x}||}\right] = \pi\left(\tfrac{\mathbf{x}}{||\mathbf{x}||}\right) \in \pi(\mathbb{S}^2);$$

hence $P_2\mathbb{R} \subseteq \pi(\mathbb{S}^2)$. Conversely, $\mathbb{S}^2 \subseteq \mathbb{R}^3 \backslash \{\mathbf{0}\}$ and so $\pi(\mathbb{S}^2) \subseteq P_2\mathbb{R}$. Thus $P_2\mathbb{R} = \pi(\mathbb{S}^2)$, and $P_2\mathbb{R}$ is compact. □

Since the product of two Hausdorff spaces is Hausdorff, and the subspace of a Hausdorff space is Hausdorff ([1], Theorem 2.6.10), we obtain the following.

Corollary 2.16. *The subspace* $(P_2\mathbb{R})^2_*$ *of* $P_2\mathbb{R} \times P_2\mathbb{R}$ *is Hausdorff.*

Let us proceed to derive an analogue of Lemma 2.9.

Lemma 2.17. *If the flag space is closed in* $P_2\mathbb{R} \times \mathcal{L}_2\mathbb{R}$, *then the joining map* $\vee : (P_2\mathbb{R})^2_* \to \mathcal{L}_2\mathbb{R}$ *is continuous.*

Proof. By Lemma 2.9, using Proposition 2.15 and Corollary 2.16, it suffices to show that Graph($\vee$) is closed in $(P_2\mathbb{R})^2_* \times \mathcal{L}_2\mathbb{R}$. Since, by the hypothesis, $\mathbf{F}$ is closed in $P_2\mathbb{R} \times \mathcal{L}_2\mathbb{R}$, the set $P_2\mathbb{R} \times \mathbf{F}$ is closed in the product space $(P_2\mathbb{R})^2 \times \mathcal{L}_2\mathbb{R}$. Now, by the homeomorphism

$$\gamma : (P_2\mathbb{R})^2 \times \mathcal{L}_2\mathbb{R} \to (P_2\mathbb{R})^2 \times \mathcal{L}_2\mathbb{R} : (\mathbf{p}, \mathbf{q}, L) \to (\mathbf{q}, \mathbf{p}, L),$$

the subset

$$\{(\mathbf{p}, \mathbf{q}, L) : \mathbf{q} \in P_2\mathbb{R}, (\mathbf{p}, L) \in \mathbf{F}\} = \gamma(P_2\mathbb{R} \times \mathbf{F})$$

is also closed in $(P_2\mathbb{R})^2 \times \mathcal{L}_2\mathbb{R}$. Thus we find that

$$\begin{aligned}
\text{Graph}(\vee) &= \Big\{\big((\mathbf{p},\mathbf{q}),L\big) : (\mathbf{p},\mathbf{q}) \in (P_2\mathbb{R})^2_*,\ \{(\mathbf{p},L),(\mathbf{q},L)\} \subseteq \mathbf{F}\Big\} \\
&= \big\{\big((\mathbf{p},\mathbf{q}),L\big) : (\mathbf{p},\mathbf{q}) \in (P_2\mathbb{R})^2_*,\ L \in \mathcal{L}_2\mathbb{R}\big\} \\
&\quad \cap \big\{\big((\mathbf{p},\mathbf{q}),L\big) : \mathbf{q} \in P_2\mathbb{R},\ (\mathbf{p},L) \in \mathbf{F}\big\} \cap \big\{\big((\mathbf{p},\mathbf{q}),L\big) : \mathbf{p} \in P_2\mathbb{R},\ (\mathbf{q},L) \in \mathbf{F}\big\} \\
&= ((P_2\mathbb{R})^2_* \times \mathcal{L}_2\mathbb{R}) \cap \underbrace{\Big(\gamma(P_2\mathbb{R} \times \mathbf{F}) \cap (P_2\mathbb{R} \times \mathbf{F})\Big)}_{\text{Closed in } (P_2\mathbb{R})^2 \times \mathcal{L}_2\mathbb{R}}
\end{aligned}$$

is closed in $(P_2\mathbb{R})^2_* \times \mathcal{L}_2\mathbb{R}$. □

[2] $||\cdot||$ denotes the Euclidean norm.

Lemma 2.18. *Suppose that X, Y are spaces and that $f : X \to Y$ is an open map. If $V \subseteq Y$ and $f^{-1}(V)$ is closed in X, then V is closed in Y.*

Proof. We have

$$Y \backslash V = f\big(\underbrace{Y \backslash \overbrace{f^{-1}(V)}^{\text{closed}}}_{\text{open}}\big)$$

is open in Y, so V is closed in Y. $\square$

Theorem 2.19. *The joining and intersection maps of the real projective plane*

$$\vee : (P_2\mathbb{R})^2_* \to \mathcal{L}_2\mathbb{R} : (\mathbf{p}, \mathbf{q}) \mapsto \mathbf{p} \vee \mathbf{q}, \text{ and}$$
$$\wedge : (\mathcal{L}_2\mathbb{R})^2_* \to P_2\mathbb{R} : (L, M) \mapsto L \wedge M$$

are continuous.

Proof. We show that the joining map is continuous; continuity of the intersection map then follows by Theorem 2.8. By Lemma 2.17, it suffices to show that $\mathbf{F}$ is closed. We first note that since the quotient maps π and λ are open (Lemma 2.7), so is the map

$$\pi \times \lambda : (\mathbb{R}^3 \backslash \{\mathbf{0}\})^2 \to P_2\mathbb{R} \times \mathcal{L}_2\mathbb{R}.$$

Hence to show that $\mathbf{F}$ is closed in $P_2\mathbb{R} \times \mathcal{L}_2\mathbb{R}$ it suffices to show that $(\pi \times \lambda)^{-1}(\mathbf{F})$ is closed in $(\mathbb{R}^3 \backslash \{\mathbf{0}\})^2$ (Lemma 2.18). We have

$$\begin{aligned}(\pi \times \lambda)^{-1}(\mathbf{F}) &= \{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^3 \backslash \{\mathbf{0}\})^2 : [\mathbf{x}] \subseteq [\mathbf{y}]^\perp\} \\ &= \{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^3 \backslash \{\mathbf{0}\})^2 : \mathbf{x} \cdot \mathbf{y} = 0\},\end{aligned}$$

which is closed in $(\mathbb{R}^3 \backslash \{\mathbf{0}\})^2$ by the continuity of the dot product. $\square$

Recalling Theorem 2.4, $\mathbb{A}_2\mathbb{R} \cong P_2\mathbb{R} \backslash W$, by restricting the line set to the subset of intersecting lines, we obtain the analogous result.

Corollary 2.20. *The joining and intersection maps of the real affine plane are continuous.*

3 Moulton Planes

We now look at a more elaborate example of an affine plane. This time we seek a topology on the line set with which the join and intersection maps are continuous; we do not have a topology inherent in its structure as we did in the real affine plane situation through its projective completion.

3.1 Definition

For $k \in \mathbb{Z}_{\geq 1}$, we define the *k-Moulton plane* $\mathcal{M}_k$ as follows. We take the incidence structure that is the real affine plane $\mathbb{A}_2\mathbb{R}$ (so the point set is $\mathbb{R}^2$) and replace the lines in $\mathfrak{L}(\mathbb{A}_2\mathbb{R})$ with negative slope s by the *kinked* lines

$$M^*_{[s,t]} = \{(x, sx + t) : x \geq 0\} \cup \{(x, skx + t) : x \leq 0\}.$$

Note that $\mathcal{M}_k \cong \mathcal{M}_{\frac{1}{k}}$ via the isomorphism $(x, y) \mapsto (-x, y)$, and $\mathcal{M}_1$ is just $\mathbb{A}_2\mathbb{R}$.

We shall use the term *Moulton plane* to refer to an arbitrary *k-Moulton plane*; we denote the line set $\mathfrak{L}(\mathcal{M}_k)$ by $\mathcal{L}$. It is not difficult to verify that a Moulton plane is an affine plane. More important for our purposes are the following properties of Moulton planes.

Theorem 3.1.

(*i*) *Each line is closed in the topological space* $\mathbb{R}^2$ *and is homeomorphic to* $\mathbb{R}$.

(*ii*) *Any two distinct points* $\mathbf{p}$ *and* $\mathbf{q}$ *lie on exactly one line* $L \in \mathcal{L}$.

Proof.
(*i*) For a vertical line $L_{[c]}$, the mapping $(c, y) \mapsto (y, 0)$ is a homeomorphism of $\mathbb{R}^2$ mapping $L_{[c]}$ onto $\mathbb{R} \times \{0\}$; for a non-vertical line, the mapping $(x, y(x)) \mapsto (x, 0)$ also performs this task. Hence embedding a line $L \in \mathcal{L}$ is topologically equivalent to the inclusion of the closed subset $\mathbb{R} \times \{0\}$ in $\mathbb{R}^2$; it also immediately follows that each line is homeomorphic to $\mathbb{R}$.
(*ii*) Let $\mathbf{p} = (x_1, y_1)$, $\mathbf{q} = (x_2, y_2)$ be distinct points in $\mathbb{R}^2$. If $x_1 = x_2$, then $\mathbf{p}$ and $\mathbf{q}$ lie uniquely on $L_{[x_1]}$. Suppose that $x_1 \neq x_2$. If $x_1, x_2 \leq 0$, then $\mathbf{p}$ and $\mathbf{q}$ lie on $L_{[s,t]}$ if $y_1 \leq y_2$ or $M^*_{[s,t]}$ otherwise, where s and t are determined by the system

$$\begin{aligned} y_1 &= sx_1 + t; \\ y_2 &= sx_2 + t. \end{aligned}$$

This has a unique solution for $\{s, t\}$ since

$$\det \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} = x_1 - x_2 \neq 0. \qquad \square$$

On the other hand, assume (without loss of generality) that $x_1 \leq 0$ and $x_2 > 0$. Then $\mathbf{p}$ and $\mathbf{q}$ lie on $L_{[s,t]}$ if $y_1 \leq y_2$ or $M^*_{[s,t]}$ otherwise, where the system

$$\begin{aligned} y_1 &= sx_1 + t; \\ y_2 &= ksx_2 + t, \end{aligned}$$

uniquely determines $\{s, t\}$ since

$$\det \begin{pmatrix} x_1 & 1 \\ kx_2 & 1 \end{pmatrix} = x_1 - kx_2 < 0.$$

For each $L \in \mathcal{L}$, from Theorem 3.1(i) we obtain that its complement $\mathbb{R}^2 \backslash L \approx \mathbb{R}^2 \backslash (\mathbb{R} \times \{0\})$ has precisely two (open) components; each is called an *open half-plane.* If H is an open half-plane corresponding to a line L, then its closure is $\overline{H} = H \cup L$ and is called a *closed half-plane.* The terms *upper* (*left-hand*) and *lower* (*right-hand*) may be used in the obvious way to distinguish between the two open/closed half-planes defined by a non-vertical (vertical) line.

For distinct points $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in $\mathbb{R}^2$ we define the *closed interval* $[\mathbf{a}, \mathbf{b}]$ as the closed path along $\mathbf{a} \vee \mathbf{b}$ from $\mathbf{a}$ to $\mathbf{b}$. We write $[\mathbf{a}, \mathbf{b}]$ under the (safe in all our considerations) assumption that $a_1 \leq b_1$, so such an interval is assumed to trace from left to right (or from bottom to top) in the plane. Thus, for the verticals lines: $[\mathbf{a}, \mathbf{b}] = (\mathbb{R} \times [a_1, b_2]) \cap (\mathbf{a} \vee \mathbf{b})$; for the non-verticals lines: $[\mathbf{a}, \mathbf{b}] = ([a_1, b_1] \times \mathbb{R}) \cap (\mathbf{a} \vee \mathbf{b})$. The *open interval* $]\mathbf{a}, \mathbf{b}[$ is defined analogously, so that $]\mathbf{a}, \mathbf{b}[= [\mathbf{a}, \mathbf{b}] \backslash \{\mathbf{a}, \mathbf{b}\}$.

The notation $[\mathbf{x}]$ for the span of a vector will not be used in this section.

3.2 Convexity Properties

In this section we introduce the notion of convexity with respect to Moulton lines. We establish how Moulton lines intersect, and we show that the point set $\mathbb{R}^2$ has a basis of "convex quadrangles".

A subset S of $\mathbb{R}^2$ is called *convex* if for each pair of distinct points $\{\mathbf{a}, \mathbf{b}\}$ in S, we have that $[\mathbf{a}, \mathbf{b}] \subseteq S$.

Theorem 3.2. *The open and closed half-planes are convex.*

Proof. For the verticals and un-kinked lines, the portion of the straight line in the Euclidean sense connecting two points in a closed half-plane is the portion of the Moulton line through those points. Hence we are only left to prove the convexity of open and closed half-planes corresponding to kinked lines.

Consider the kinked line $L := M^*_{[s,t]}$ and the corresponding upper open half-plane H. Let $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in H$, so

$$a_2 > sa_1 + t, \tag{3.2.1}$$

$$b_2 > sb_1 + t. \tag{3.2.2}$$

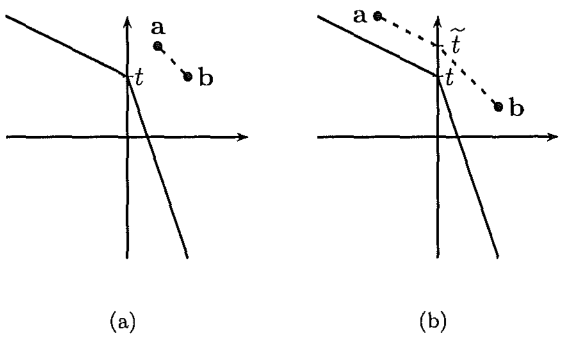


Figure 3.1

If $\mathbf{a}$ and $\mathbf{b}$ lie on the same side of the y-axis then $[\mathbf{a}, \mathbf{b}]$ lies in H by the convexity of the Euclidean half-plane (see Figure 3.1(a)). Now suppose that $\mathbf{a}$ and $\mathbf{b}$ are such that $a_1 < 0 < b_1$. To show that $[\mathbf{a}, \mathbf{b}]$ does not intersect L it suffices to show that the intercept of $\mathbf{a} \vee \mathbf{b} := M^*_{[\tilde{s},\tilde{t}]}$ is strictly greater than t (see Figure 3.1(b)). From

$$\begin{cases} a_2 &= \tilde{s}a_1 + \tilde{t} \\ b_2 &= k\tilde{s}b_1 + \tilde{t} \end{cases}$$

we obtain that

$$\tilde{t} = \frac{b_1 a_2 - \frac{1}{k} a_1 b_2}{b_1 - \frac{1}{k} a_1}.$$

Now, since $b_1 > 0$ and $-\frac{1}{k}a_1 > 0$, (3.2.1) and (3.2.2) imply, respectively, that $b_1 a_2 > sa_1 b_1 + tb_1$ and $-\frac{1}{k}a_1 b_2 > -sa_1 b_1 - \frac{1}{k}ta_1$. Thus,

$$b_1 a_2 - \tfrac{1}{k} a_1 b_2 > t\underbrace{(b - \tfrac{1}{k} a_1)}_{>0}$$

and hence

$$t < \frac{b_1 a_2 - \frac{1}{k} a_1 b_2}{b - \frac{1}{k} a_1} = \tilde{t}.$$

The lower open half-plane corresponding to L may be considered analogously. Convexity of the closed half-planes follows by replacing the (strict) inequalities in the above proof with their non-strict analogues. $\square$

We say that two points lie on the *same side* (*different* or *opposite side*) of a line L if they belong to the same open half-plane (different open half-planes) corresponding to L. With the notion of convexity now at hand we can prove some results concerning line intersections.

Corollary 3.3.

(*i*) *Let $K, L \in \mathcal{L}$ be distinct lines intersecting at a point* $\mathbf{p}$. *Then any two distinct points* $\mathbf{a}, \mathbf{b} \in K$ *with* $\mathbf{p} \in [\mathbf{a}, \mathbf{b}]$ *lie on different sides of* L.

(*ii*) *A line $L \in \mathcal{L}$ and a closed interval* $[\mathbf{a}, \mathbf{b}]$ *with* $\mathbf{a}, \mathbf{b} \notin L$ *are disjoint if and only if* $\mathbf{a}$ *and* $\mathbf{b}$ *lie on the same side of* L.

Proof.
(*i*) As a result of Theorem 3.2, if two distinct points $\mathbf{a}$ and $\mathbf{b}$ lie on the same side of L, then the open interval $]\mathbf{a}, \mathbf{b}[$ must lie in the same open half-plane determined by L. Hence there can be no point $\mathbf{p} \in]\mathbf{a}, \mathbf{b}[$ with $\mathbf{p} \in L$.
(*ii*) Let $\mathbf{a}$ and $\mathbf{b}$ be two distinct points on the same side of L. Since $L \cap [\mathbf{a}, \mathbf{b}] = L \cap]\mathbf{a}, \mathbf{b}[$, a point in this intersection would contradict (*i*). Conversely, since a line $L \in \mathcal{L}$ embeds in $\mathbb{R}^2$ homeomorphically to $\mathbb{R} \times \{0\}$ (Theorem 3.1), it separates the plane $\mathbb{R}^2$, so if $\mathbf{a}$ and $\mathbf{b}$ are points which lie on different sides of L, then the connected set $[\mathbf{a}, \mathbf{b}]$ must intersect L. □

Define the *convex hull* $\langle \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \rangle$ of the n distinct points $\mathbf{a}_i$ by

$$\langle \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \rangle = \overline{H_1} \cap \overline{H_2} \cap \ldots \cap \overline{H_n},$$

where $\overline{H_i}$ is closed half-plane determined by $\mathbf{a}_j \vee \mathbf{a}_k$ containing $\mathbf{a}_i$, for $i, j, k \in \{1, \ldots, n\}$. Since the intersection of convex sets is convex, from Theorem 3.2 we have that a convex hull and its interior are convex.

Corollary 3.4. *Let* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ *be three non-collinear points. Then*

(*i*) *Every line L meeting* $[\mathbf{a}_1, \mathbf{a}_2] \cup [\mathbf{a}_2, \mathbf{a}_3] \cup [\mathbf{a}_3, \mathbf{a}_1]$ *intersects at least two of the sides* $[\mathbf{a}_i, \mathbf{a}_j]$.

(*ii*) *A line L meets the interior of* $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle$ *if and only if L meets one of* $]\mathbf{a}_i, \mathbf{a}_j[$ *with* $L \neq \mathbf{a}_i \vee \mathbf{a}_j$.

Proof.
(*i*) If L intersects $[\mathbf{a}_1, \mathbf{a}_2] \cup [\mathbf{a}_2, \mathbf{a}_3] \cup [\mathbf{a}_3, \mathbf{a}_1]$ at one of the vertices, $\mathbf{a}_1$ say, then it intersects both $[\mathbf{a}_1, \mathbf{a}_2]$ and $[\mathbf{a}_3, \mathbf{a}_1]$. Now suppose that $L \cap \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} = \varnothing$ and without loss of generality assume that there is a point $\mathbf{p} = L \wedge]\mathbf{a}_1, \mathbf{a}_2[$. Then, by Corollary 3.3(i), $\mathbf{a}_1$ and $\mathbf{a}_2$ lie on opposite sides of L. Without loss of generality, since $\mathbf{a}_3$ is not on L, assume $\mathbf{a}_2$ and $\{\mathbf{a}_1, \mathbf{a}_3\}$ lie on opposite sides of L. As $\mathbf{a}_2$ and $\mathbf{a}_3$ lie on opposite sides of L, it follows from Corollary 3.3(ii) that L intersects $[\mathbf{a}_2, \mathbf{a}_3]$ as well.
(*ii*) Let $Q := \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle$. Suppose that L meets $\operatorname{int}(Q)$ at a point $\mathbf{p}$. Then $L \backslash \{\mathbf{p}\} \approx \mathbb{R} \backslash \{0\}$ has precisely two components — call them L_1 and L_2 — which must not be contained in the bounded set Q. Being connected, L_1 and L_2 each intersect $\partial Q := \overline{Q} \backslash Q = [\mathbf{a}_1, \mathbf{a}_2] \cup [\mathbf{a}_2, \mathbf{a}_3] \cup [\mathbf{a}_3, \mathbf{a}_1]$; let $\mathbf{q}_1 = \overline{L_1} \cap \partial Q$ and $\mathbf{q}_2 = \overline{L_2} \cap \partial Q$. Since $\mathbf{p}$ is an interior point of Q, we have that $\mathbf{p} \notin \{\mathbf{q}_1, \mathbf{q}_2\}$ and so, as $\overline{L_1} \cap \overline{L_2} = \{\mathbf{p}\}$, $\mathbf{q}_1 \neq \mathbf{q}_2$. Furthermore, if $\{\mathbf{q}_1, \mathbf{q}_2\} \subseteq \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, then $L = \mathbf{q}_1 \vee \mathbf{q}_2$ does not contain an interior point, so without loss of generality assume that $\mathbf{q}_1 \notin \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Thus, $\mathbf{q}_1 \in \partial Q \backslash \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and so $\mathbf{q}_1$ is an intersection point of L and some $]\mathbf{a}_i, \mathbf{a}_j[$. Finally, note that since, by (*i*), L intersects another side of ∂Q, if $L = \mathbf{a}_i \vee \mathbf{a}_j$ then $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are collinear — a contradiction — so we have that $L \neq \mathbf{a}_i \vee \mathbf{a}_j$.

Conversely, suppose that a line $L \neq \mathbf{a}_1 \vee \mathbf{a}_2$ intersects $]\mathbf{a}_1, \mathbf{a}_2[$ at a point $\mathbf{q}$. By (*i*), L intersects another side, say $[\mathbf{a}_2, \mathbf{a}_3]$. Let $\mathbf{q}' = L \cap [\mathbf{a}_2, \mathbf{a}_3] \neq \mathbf{q}$, so that $L = \mathbf{q} \vee \mathbf{q}'$. Now, $[\mathbf{q}, \mathbf{q}']$ is contained in the convex set Q but not in ∂Q, so $[\mathbf{q}, \mathbf{q}'] \subseteq \operatorname{int}(Q)$. Thus L meets the interior of Q. The other cases are analogous. □

A *convex triangle* is the convex hull of three non-collinear points; a *convex quadrangle* is the convex hull of four points, no three of which are collinear.

Lemma 3.5. *Let* $Q := \langle \mathbf{a}_i, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle$ *be a convex quadrangle and let* $\mathbf{x}_i \in]\mathbf{a}_i, \mathbf{a}_{i+1}[$ *for* $1 \leq i \leq 4$, *where* $\mathbf{a}_5 = \mathbf{a}_1$. *Then the intervals* $[\mathbf{x}_1, \mathbf{x}_3]$ *and* $[\mathbf{x}_2, \mathbf{x}_4]$ *intersect at an interior point of Q. (See Figure 3.2)*

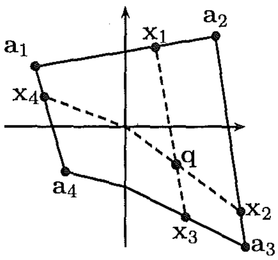


Figure 3.2

Proof. Let $L = \mathbf{x}_2 \vee \mathbf{x}_4$; then L intersects $[\mathbf{a}_2, \mathbf{a}_3]$ at $\mathbf{x}_2$, so $\mathbf{a}_2$ and $\mathbf{a}_3$ lie on opposite sides of L (Corollary 3.3(i)). Similarly, $\mathbf{a}_1$ and $\mathbf{a}_4$ lie on opposite sides of L. Now, if $\mathbf{a}_1$ and $\mathbf{a}_2$ lie on different sides of L then Corollary 3.3(i) implies that L intersects $[\mathbf{a}_1, \mathbf{a}_2]$ at some point $\mathbf{p}$. But then $\mathbf{x}_2$, $\mathbf{x}_4$ and $\mathbf{p}$ are distinct points on L, so one point lies in the open interval between the other two — contradicting the convexity of $\mathrm{int}(Q)$. Hence the sets $\{\mathbf{a}_1, \mathbf{a}_2\}$ and $\{\mathbf{a}_3, \mathbf{a}_4\}$ lie on opposite sides of L. By the convexity of the respective open half-planes determined by L we thus obtain that $\mathbf{x}_1 \in]\mathbf{a}_1, \mathbf{a}_2[$ and $\mathbf{x}_3 \in]\mathbf{a}_3, \mathbf{a}_4[$ also lie in these open half-planes. Thus $\mathbf{x}_1$ and $\mathbf{x}_3$ lie on opposite sides of L and $[\mathbf{x}_1, \mathbf{x}_3] \cap L \neq \varnothing$ (Corollary 3.3(ii)); let $\mathbf{q} \in [\mathbf{x}_1, \mathbf{x}_3] \cap L$.

Analogously, letting $L' = \mathbf{x}_1 \vee \mathbf{x}_4$, we find a point $\mathbf{q}' \in [\mathbf{x}_2, \mathbf{x}_4] \cap L'$. Now, if $\mathbf{q} \neq \mathbf{q}'$ then L and L' are distinct lines joining the two points, contradicting Theorem 3.1(ii). Thus $\mathbf{q}$ is the intersection point of $[\mathbf{x}_1, \mathbf{x}_3]$ and $[\mathbf{x}_2, \mathbf{x}_4]$. Finally, since the $\mathbf{x}_i$ $(i = 1, 2, 3, 4)$ are mutually distinct (for instance, $\mathbf{x}_1$ and $\mathbf{x}_3$ lie on opposite sides of a line on which $\mathbf{x}_2$ lies), we have that $\mathbf{q} \in]\mathbf{x}_1, \mathbf{x}_3[\subseteq \mathrm{int}(Q)$ by the convexity of $\mathrm{int}(Q)$. □

A collection $\mathfrak{N}$ of subsets of $\mathbb{R}^2$ each containing the point $\mathbf{p}$ is called a *neighbourhood basis* for $\mathbf{p}$ if for every open neighbourhood $U \subseteq \mathbb{R}^2$ of $\mathbf{p}$, there is a $V \in \mathfrak{N}$ contained in U.

Lemma 3.6. *Every point $\mathbf{p} \in \mathbb{R}^2$ has a neighbourhood basis consisting of finite intersections of open half-planes.*

Proof. Let $U \subseteq \mathbb{R}^2$ be an open neighbourhood of $\mathbf{p}$. In $\mathbb{R}^2$ there is an open ball[3] $D := B(\mathbf{p}, r)$ for some $r > 0$ contained in U. For each $\mathbf{x} \in \partial D := \overline{D} \backslash D$, let $L_\mathbf{x}$ be a line intersecting $\mathbf{x} \vee \mathbf{p}$ at precisely one point $\mathbf{q} \in]\mathbf{p}, \mathbf{x}[$, so by Corollary 3.3(i) we have that $\mathbf{p}$ and $\mathbf{x}$ lie on opposite sides of $L_\mathbf{x}$. Let $G_\mathbf{x}$ and $H_\mathbf{x}$ be the corresponding open half-planes containing $\mathbf{p}$ and $\mathbf{x}$, respectively (see Figure 3.3).

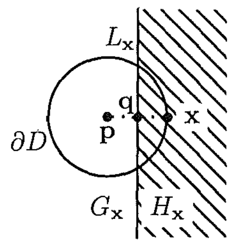


Figure 3.3

Since

$$\partial D \subseteq \bigcup_{\mathbf{x} \in \partial D} H_\mathbf{x}$$

[3]The *open ball* with *centre* $\mathbf{x}$ and *radius* $r > 0$ in $\mathbb{R}^2$ is defined by

$$B(\mathbf{x}, r) := \{\mathbf{y} \in X : ||\mathbf{x} - \mathbf{y}|| < r\}.$$

and ∂D is compact, there is a finite N number of $\mathbf{x}_i \in \partial D$ such that $\partial D \subseteq \bigcup_{i=1}^N H_{\mathbf{x}_i}$. We show that the corresponding intersection $\bigcap_{i=1}^N G_{\mathbf{x}_i}$ is contained in D (and hence U). Since each $G_{\mathbf{x}_i}$ is convex (Theorem 3.2) and the finite intersection of convex sets is convex, $\bigcap_{i=1}^N G_{\mathbf{x}_i}$ is convex. Now, convexity with respect to Moulton lines implies path connectedness, so $\bigcap_{i=1}^N G_{\mathbf{x}_i}$ is connected. Noting that if $\mathbf{y} \in \bigcap_{i=1}^N G_{\mathbf{x}_i}$ then $\mathbf{y} \notin \bigcup_{i=1}^N H_{\mathbf{x}_i}$, we have that

$$\bigcap_{i=1}^N G_{\mathbf{x}_i} \cap \partial D \subseteq \bigcap_{i=1}^N G_{\mathbf{x}_i} \cap \bigcup_{i=1}^N H_{\mathbf{x}_i} = \varnothing,$$

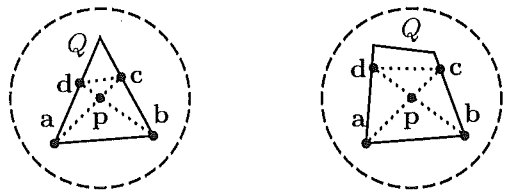
and thus the connected set $\bigcap_{i=1}^N G_{\mathbf{x}_i}$ must lie entirely in D.

The collection of such finite intersections of open half-planes, each dependent on an open neighbourhood of $\mathbf{p}$, thus provide a neighbourhood basis for $\mathbf{p}$. □

Theorem 3.7. *Every point $\mathbf{p} \in \mathbb{R}^2$ has a neighbourhood basis consisting of convex quadrangles or of convex triangles.*

Proof. Following Lemma 3.6, each (bounded) open ball in $\mathbb{R}^2$ with centre $\mathbf{p}$ contains a finite intersection of open half-planes. Since this intersection must be bounded, it consists of three open half-planes intersecting to form a convex triangle, or it includes four different open half-planes, whose intersection forms a convex quadrangle.

Given a convex triangle or convex quadrangle Q with interior point $\mathbf{p}$, labelling two of the vertices of Q by $\mathbf{a}$ and $\mathbf{b}$, and letting $\mathbf{c}$ and $\mathbf{d}$ be the intersection points of $\mathbf{a} \vee \mathbf{p}$ and $\mathbf{b} \vee \mathbf{p}$ with Q, respectively, (see Figure 3.4) we have that $\mathbf{p}$ is in the interior of $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle$, where $\mathbf{p} = (\mathbf{a} \vee \mathbf{c}) \wedge (\mathbf{b} \vee \mathbf{d})$.



(a) Convex triangle (b) Convex quadrangle

Figure 3.4

□

3.3 The Natural Topology

Denote the set of distinct points

$$(\mathbb{R}^2 \times \mathbb{R}^2)_* := (\mathbb{R}^2 \times \mathbb{R}^2) \backslash \{(\mathbf{p}, \mathbf{p}) : \mathbf{p} \in \mathbb{R}^2\}$$

and the set of intersecting lines

$$\mathcal{L}_*^2 := \{(K, L) \in \mathcal{L}^2 : K \cap L \neq \varnothing,\, K \neq L\}.$$

The collinearity of triples of points and their order is preserved under limits in the following sense.

Proposition 3.8.

(*i*) *Let $(\mathbf{a}_i)$, $(\mathbf{b}_i)$ and $(\mathbf{c}_i)$ be sequences in $\mathbb{R}^2$ converging to mutually distinct points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, and suppose that $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{c}_i$ are collinear for infinitely many i. Then $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are collinear.*

(*ii*) *In addition to the hypotheses of (i), suppose that $\mathbf{b}_i \in \,]\mathbf{a}_i, \mathbf{c}_i[$ for infinitely many i. Then $\mathbf{b} \in \,]\mathbf{a}, \mathbf{c}[$.*

Proof.
(i) If $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{c}_i$ are not mutually distinct for infinitely many i, then without loss of generality there are subsequences $(\mathbf{a}_{i_k})$ and $(\mathbf{b}_{i_k})$ of $(\mathbf{a}_i)$ and $(\mathbf{b}_i)$, respectively, such that $\mathbf{a}_{i_k} = \mathbf{b}_{i_k}$ for all k — contradicting that their limit points are distinct. Hence we may choose subsequences and relabel them as $(\mathbf{a}_i)$, $(\mathbf{b}_i)$ and $(\mathbf{c}_i)$ so that

$$\mathbf{b}_i \in]\mathbf{a}_i, \mathbf{c}_i[\text{ for all } i \in \mathbb{N}. \tag{$*$}$$

Suppose that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are *not* collinear. Let K be a line intersecting $]\mathbf{a}, \mathbf{b}[$ and $]\mathbf{b}, \mathbf{c}[$ (Figure 3.5). Then $\mathbf{b}$ and $\mathbf{a}$, and $\mathbf{b}$ and $\mathbf{c}$, lie on opposite sides of K (Corollary 3.3(ii)), so $\mathbf{a}$ and $\mathbf{c}$ lie on the same side of K. Hence (again by Corollary 3.3(ii)) $]\mathbf{a}, \mathbf{c}[$ and K are disjoint, so $\mathbf{b}$ and $]\mathbf{a}, \mathbf{c}[$ must lie on opposite sides of K. Now, if there were subsequences $(\mathbf{a}_{i_k})$, $(\mathbf{b}_{i_k})$ and $(\mathbf{c}_{i_k})$ such that $\mathbf{b}_{i_k}$ and $[\mathbf{a}_{i_k}, \mathbf{c}_{i_k}]$ lie on the same side of K for all k, then their limit points would belong to the same closed half-plane determined by K — contradicting that $\mathbf{b}$ and $[\mathbf{a}, \mathbf{c}]$ lie on opposite sides of K. Hence there is an $N > 0$ such that $\mathbf{b}_i$ and $[\mathbf{a}_i, \mathbf{c}_i]$ lie on opposite sides of K for all $i > N$ — contradicting ($*$). We deduce that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ must be collinear.

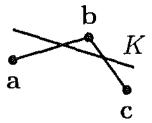


Figure 3.5

(ii) Suppose, on the contrary, that $\mathbf{b} \notin]\mathbf{a}, \mathbf{c}[$. By ($i$), $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are collinear, so either $\mathbf{c} \in]\mathbf{a}, \mathbf{b}[$ or $\mathbf{a} \in]\mathbf{b}, \mathbf{c}[$. By symmetry it suffices to consider the first case only: suppose that $\mathbf{c} \in]\mathbf{a}, \mathbf{b}[$. Let K be a line not equal to $\mathbf{a} \vee \mathbf{b}$ which intersects $]\mathbf{c}, \mathbf{b}[$, so that $\mathbf{b}$ and $\mathbf{c}$ lie on opposite sides of K (see Figure 3.6). If $[\mathbf{a}_i, \mathbf{c}_i]$ and $\mathbf{b}_i$ are on the same side of K for infinitely many i, then the limit points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ of the corresponding subsequences $(\mathbf{a}_{i_k})$, $(\mathbf{b}_{i_k})$ and $(\mathbf{c}_{i_k})$ lie in the same closed half-plane determined by K — contradicting the choice of K. Hence there is an $N > 0$ such that $[\mathbf{a}_i, \mathbf{c}_i]$ and $\mathbf{b}_i$ are on opposite sides of K for all $i > N$, which contradicts the hypothesis that $\mathbf{b}_i \in]\mathbf{a}_i, \mathbf{c}_i[$ for infinitely many i. We deduce that $\mathbf{b} \in]\mathbf{a}, \mathbf{c}[$.

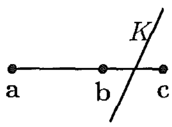


Figure 3.6

□

Our first step to defining a suitable topology on the line set $\mathcal{L}$ is to introduce a notion of convergence so that if a sequence of pairs of points in $\mathbb{R}^2$ converges to a (distinct) pair, then the sequence of lines determined by these pairs converge correspondingly. Let $(L_i) \in \mathcal{L}^\infty$. We write $(\mathbf{p}_i) \in (L_i)$ to indicate that $(\mathbf{p}_i)$ is a sequence in $\mathbb{R}^2$ such that $\mathbf{p}_i \in L_i$ for each i. We denote the set of all the limit points of all the sequences $(\mathbf{p}_i) \in (L_i)$ by $\liminf_i L_i$, and the set of all the accumulation points of all the sequences $(\mathbf{p}_i) \in (L_i)$ by $\limsup_i L_i$. That is,

$$\liminf_i L_i := \left\{\mathbf{p} : \mathbf{p} = \lim_{i \to \infty} \mathbf{p}_i \text{ for some convergent } (\mathbf{p}_i) \in (L_i)\right\}$$

and

$$\limsup_i L_i := \{\mathbf{p} : \mathbf{p} = \lim_{k \to \infty} \mathbf{p}_{i_k} \text{ for some convergent subsequence } (\mathbf{p}_{i_k}) \text{ of some } (\mathbf{p}_i) \in (L_i)\}.$$

We define *Hausdorff convergence* as follows. We say that a sequence $(L_i) \in \mathcal{L}^\infty$ Hausdorff-converges to $L \in \mathcal{L}$ and write $(L_i) \xrightarrow{Haus} L$ if

$$\liminf_i L_i = L = \limsup_i L_i. \tag{†}$$

Justification: The first equality of (†) means that each point on L is the limit of some sequence $(\mathbf{p}_i) \in (L_i)$ (see Figure 3.7(a)). The second equality prevents us from selecting a sequence $(\mathbf{p}_i) \in (L_i)$ that stays away from L (see Figure 3.7(b)).

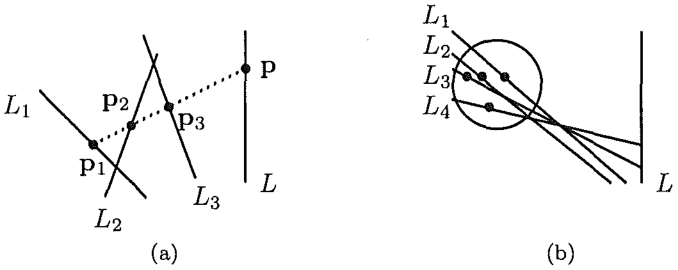


Figure 3.7

Note that if $\mathbf{p}$ is a limit point of a sequence $(\mathbf{p}_i) \in (L_i)$, then $\mathbf{p}$ is certainly an accumulation point of $(\mathbf{p}_i)$; thus $\liminf_i L_i \subseteq \limsup_i L_i$. To show that $(L_i) \xrightarrow{Haus} L$ it therefore suffices to verify that

$$\limsup_i L_i \subseteq L \subseteq \liminf_i L_i. \tag{$\star$}$$

With this notion of the convergence for lines, the join map is sequentially continuous.

Proposition 3.9. *Let $(\mathbf{a}_i)$ and $(\mathbf{b}_i)$ be sequences in $\mathbb{R}^2$ converging to distinct points $\mathbf{a}$ and $\mathbf{b}$, respectively. Then the sequence of lines (L_i), defined by $L_i = \mathbf{a}_i \vee \mathbf{b}_i$ for each i, Hausdorff-converges to $L = \mathbf{a} \vee \mathbf{b}$.*

Proof. We verify that ($\star$) holds.

- $\limsup_i L_i \subseteq L$: Let $\mathbf{c}$ be an accumulation point of some $(\mathbf{c}_i) \in (L_i)$; choose a subsequence $(\mathbf{c}_{i_k})$ of $(\mathbf{c}_i)$ converging to $\mathbf{c}$. If $\mathbf{c} \in \{\mathbf{a}, \mathbf{b}\}$ then $\mathbf{c} \in \mathbf{a} \vee \mathbf{b} = L$ and we are done, so suppose that $\mathbf{c}$, $\mathbf{a}$ and $\mathbf{b}$ are mutually distinct. Then $(\mathbf{c}_{i_k})$, $(\mathbf{a}_{i_k})$, $(\mathbf{b}_{i_k})$ are collinear (on L_{i_k}) for infinitely many k and have mutually distinct limit points. Thus by Proposition 3.8(i) the limit points $\mathbf{c}$, $\mathbf{a}$ and $\mathbf{b}$ are collinear and hence $\mathbf{c} \in \mathbf{a} \vee \mathbf{b} = L$.
- $L \subseteq \liminf_i L_i$: Let $\mathbf{c} \in L = \mathbf{a} \vee \mathbf{b}$. We find a sequence $(\mathbf{c}_i) \in (L_i)$ converging to $\mathbf{c}$.

▷ *Case 1:* $\mathbf{c} \in]\mathbf{a}, \mathbf{b}[$

Let K be a line distinct from L, intersecting L at $\mathbf{c}$. Now, if the points $\mathbf{a}_i$ and $\mathbf{b}_i$ lie in the same side of K for infinitely many i then $\mathbf{a}$ and $\mathbf{b}$ lie in the same closed half-plane determined by K — contradicting Corollary 3.3(ii), as K intersects $[\mathbf{a}, \mathbf{b}]$ at $\mathbf{c}$. Therefore there is an $N > 0$ such that $\mathbf{a}_i$ and $\mathbf{b}_i$ lie on opposite sides of K for all $i > N$. By relabelling if necessary define the corresponding sequence (c_i) by $\mathbf{c}_i = (\mathbf{a}_i \vee \mathbf{b}_i) \wedge K$.

We now show that any accumulation point of $(\mathbf{c}_i)$ must equal $\mathbf{c}$. Let $\widetilde{\mathbf{c}}$ be an accumulation point of $(\mathbf{c}_i)$. Since $\widetilde{\mathbf{c}} \in K$, we have that $\widetilde{\mathbf{c}} \notin \{\mathbf{a}, \mathbf{b}\}$, so $\widetilde{\mathbf{c}}$, $\mathbf{a}$ and $\mathbf{b}$ are mutually distinct. Hence $(\mathbf{a}_i)$, $(\mathbf{b}_i)$ and $(\widetilde{\mathbf{c}}_i)$ are collinear for each i (on $\mathbf{a}_i \vee \mathbf{b}_i$) and converge to distinct limits, so by Corollary 3.3(ii) we have that $\widetilde{\mathbf{c}} \in \mathbf{a} \vee \mathbf{b}$. Moreover, since K is closed we have that $\widetilde{\mathbf{c}} \in K$ and thus $\widetilde{\mathbf{c}} = (\mathbf{a} \vee \mathbf{b}) \wedge K = \mathbf{c}$.

It remains to show that such an accumulation point of $(\mathbf{c}_i)$ exists. By Theorem 3.7 there are convex quadrangles (or triangles) $Q_\mathbf{a}$ and $Q_\mathbf{b}$ containing $\mathbf{a}$ and $\mathbf{b}$, respectively, in the respective sides of K (see Figure 3.8). Since $(\mathbf{a}_i) \to \mathbf{a}$ and $(\mathbf{b}_i) \to \mathbf{b}$, there is an $N > 0$ such that $\mathbf{a}_i \in Q_\mathbf{a}$ and $\mathbf{b}_i \in Q_\mathbf{b}$ for all

$i > N$. Hence $\mathbf{c}_i$ lies in the compact set $K \cap (\cup\{[\mathbf{x},\mathbf{y}] : \mathbf{x} \in Q_\mathbf{a}, \mathbf{y} \in Q_\mathbf{b}\})$ for all $i > N$, so $(\mathbf{c}_i)$ has an accumulation point.

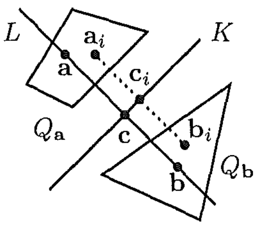


Figure 3.8

▷ *Case 2:* $\mathbf{c} \in (\mathbf{a} \vee \mathbf{b}) \backslash [\mathbf{a}, \mathbf{b}]$

Without loss of generality we may assume that $\mathbf{b} \in]\mathbf{a}, \mathbf{c}[$. By removing the lines in (L_i) equal to L (if $L_i = L$ for every i then the proposition holds trivially) and relabelling, assume that $L_i \neq L$ for all i. Now, if $L_i \wedge L \in]\text{-}\infty, \mathbf{a}[$ for all i, then $\mathbf{b} = \lim_{i \to \infty} \mathbf{b}_i \in \overline{]\text{-}\infty, \mathbf{a}[} =]\text{-}\infty, \mathbf{a}]$ — contradicting that $\mathbf{b} \in]\mathbf{a}, \mathbf{c}[$ — so we may choose a point $\mathbf{q} \in L$ such that $\mathbf{a} \in]\mathbf{q}, \mathbf{b}[$.

Now, let K be a line distinct from and intersecting L at $\mathbf{c}$, and choose points $\mathbf{r}_1$, $\mathbf{r}_2 \in K$ on opposite sides of L, so that $\mathbf{c} \in]\mathbf{r}_1, \mathbf{r}_2[$.

We first show that we cannot have $L_i \cap [\mathbf{r}_1, \mathbf{r}_2] = \varnothing$ for infinitely many i, for suppose this were the case (see Figure 3.9). By relabelling assume that L_i satisfies this equality for all i. For sufficiently large i, $\mathbf{a}_i$ and $\mathbf{b}_i$ lie in $\text{int}(\langle \mathbf{q}, \mathbf{r}_1, \mathbf{r}_2 \rangle)$, and hence by convexity we have that $]\mathbf{a}, \mathbf{b}[\subseteq \text{int}(\langle \mathbf{q}, \mathbf{r}_1, \mathbf{r}_2 \rangle)$. Thus, as these L_i do not intersect $[\mathbf{r}_1, \mathbf{r}_2]$, by Corollary 3.4 the L_i intersect $[\mathbf{q}, \mathbf{r}_1]$ and $[\mathbf{q}, \mathbf{r}_2]$.

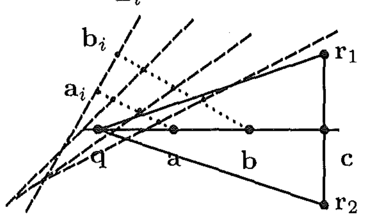


Figure 3.9

Now, for sufficiently large i, $\mathbf{c}_i \in L_i \cap ([\mathbf{q}, \mathbf{r}_1] \cup [\mathbf{q}, \mathbf{r}_2])$ lies on the boundary of $\langle \mathbf{q}, \mathbf{r}_1, \mathbf{r}_2 \rangle$, whereas $\mathbf{a}_i$ and $\mathbf{b}_i$ lie in the interior, so $\mathbf{c}_i \notin]\mathbf{a}_i, \mathbf{b}_i[$. Therefore, without loss of generality and by relabelling we may choose a sequence $(\mathbf{c}_i)$ such that $\mathbf{b}_i \in (\mathbf{a}_i, \mathbf{c}_i)$ for each i. The sequence $(\mathbf{c}_i)$ has a subsequence $(\mathbf{c}_{i_k})$ converging to a point $\widetilde{\mathbf{c}}$ in the compact set $[\mathbf{q}, \mathbf{r}_1] \cup [\mathbf{q}, \mathbf{r}_2]$. Thus $\mathbf{a}_{i_k}$, $\mathbf{b}_{i_k}$ and $\mathbf{c}_{i_k}$ are collinear for each k (on L_{i_k}) and converging to mutually distinct points $\mathbf{a}$, $\mathbf{b}$ and $\widetilde{\mathbf{c}}$. Hence Proposition 3.8(i) implies that $\mathbf{a}$, $\mathbf{b}$ and $\widetilde{\mathbf{c}}$ are collinear, so

$$\widetilde{\mathbf{c}} \in L \cap ([\mathbf{q}, \mathbf{r}_1] \cup [\mathbf{q}, \mathbf{r}_2]) = \{\mathbf{q}\};$$

that is, $\widetilde{\mathbf{c}} = \mathbf{q}$. But $\mathbf{b}_i \in]\mathbf{a}_i, \mathbf{c}_i[$ for all i, so Proposition 3.8(ii) implies that $\mathbf{b} \in]\mathbf{a}, \widetilde{\mathbf{c}}[=]\mathbf{a}, \mathbf{q}[$ — contradicting our choice of $\mathbf{q}$. We deduce there is an $N > 0$ such that $L_i \cap [\mathbf{r}_1, \mathbf{r}_2] \neq \varnothing$ for all $i > N$. By the compactness of $[\mathbf{r}_1, \mathbf{r}_2]$ and by relabelling there is a sequence $c_i := L_i \cap [\mathbf{r}_1, \mathbf{r}_2]$ converging to some point $\widetilde{\mathbf{c}} \in [\mathbf{r}_1, \mathbf{r}_2]$. Then $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{c}_i$ are collinear for each i (on L_i) and thus their mutually distinct limits $\mathbf{a}$, $\mathbf{b}$ and $\widetilde{\mathbf{c}}$ are collinear (Proposition 3.8(i)). This in turn implies that $\mathbf{r}_1$, $\widetilde{\mathbf{c}}$ and $\mathbf{r}_2$ are mutually distinct, since if $\mathbf{r}_1 = \widetilde{\mathbf{c}}$, for example, then $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}_1$ are collinear. Hence by Proposition 3.8(i), since $\mathbf{r}_1$, $\mathbf{c}_i$ and $\mathbf{r}_2$ are collinear for all i we have that $\mathbf{r}_1$, $\widetilde{\mathbf{c}}$ and $\mathbf{r}_2$ are collinear. Thus

$$\widetilde{\mathbf{c}} = (\mathbf{a} \vee \mathbf{b}) \wedge [\mathbf{r}_1, \mathbf{r}_2] = \mathbf{c}. \qquad \square$$

We are now in a position to define a topology on the line set $\mathcal{L}$ with respect to which the join and intersection maps are continuous. Define the *natural topology* H on $\mathcal{L}$ as follows. A set $A \subseteq \mathcal{L}$ is closed in H if and only if whenever a sequence $(L_i) \in A^\infty$ Hausdorff-converges to L, we have that $L \in A$. The open sets of H are precisely the complements of the H-closed subsets of $\mathcal{L}$. To verify that H is indeed a topology on $\mathcal{L}$ we first show the following familiar property of sequence convergence in a metric space holds for Hausdorff convergence.

Proposition 3.10. *A sequence $(L_i) \in \mathcal{L}^\infty$ Hausdorff-converges to $L \in \mathcal{L}$ if and only every subsequence (L_{i_k}) of (L_i) Hausdorff-converges to L.*

Proof. First suppose that $(L_i) \xrightarrow{Haus} L$ and let (L_{i_k}) be a subsequence of (L_i). It suffices to show that

$$\limsup_k L_{i_k} \subseteq L \subseteq \liminf_k L_{i_k}.$$

• $\limsup\limits_k L_{i_k} \subseteq L$: If $\mathbf{q}$ is an accumulation point of some sequence $(\mathbf{q}_{i_k}) \in (L_{i_k})$, then $\mathbf{q}$ is an accumulation point of $(\mathbf{q}_i) \in (L_i)$, so $\mathbf{q} \in \limsup\limits_i L_i = L$.

• $L \subseteq \liminf\limits_k L_{i_k}$: Let $\mathbf{q} \in L = \liminf\limits_i L_i$. Then, as $\mathbb{R}^2$ is a metric space, we have that

$$\mathbf{q} = \lim_{i \to \infty} \mathbf{q}_i = \lim_{k \to \infty} \mathbf{q}_{i_k}$$

for every subsequence $(\mathbf{q}_{i_k}) \in (L_{i_k})$. Hence $\mathbf{q} \in \liminf\limits_k L_{i_k}$.

Conversely, if every subsequence (L_{i_k}) of (L_i) Hausdorff-converges to L, then so does (L_i), being a subsequence of itself. $\square$

Theorem 3.11. *The natural topology H is a topology on $\mathcal{L}$.*

Proof. Since $\liminf \varnothing = \limsup \varnothing = \varnothing$, a sequence of empty sets Hausdorff-converges to the empty set, so $\varnothing$ is H-closed. By the definition, a Hausdorff-convergent sequence of lines Hausdorff-converges to a line, so $\mathcal{L}$ is certainly H-closed.

We now show that the finite union of H-closed sets is H-closed. By induction it suffices to consider the union of two H-closed subsets $A, B \subseteq \mathcal{L}$. Let $(L_i) \in (A \cup B)^\infty$ and suppose that $L_i \xrightarrow{Haus} L$ for some $L \in \mathcal{L}$. If $L_i \in A$ for only finitely many i then there is an $N > 0$ such that $L_i \in B$ for all $i > N$. Hence, since B is H-closed, we have that

$$L = \lim_{i \to \infty} L_i = \lim_{\substack{i \to \infty \\ i > N}} L_i \in B \subseteq A \cup B.$$

Now suppose there are infinitely many i for which $L_i \in A$ and $L_i \in B$, respectively. Let $(L_{i_k}) \in A^\infty$ and $(L_{i_l}) \in B^\infty$ be the corresponding subsequences of (L_i). By Proposition 3.10, we have that $(L_{i_k}) \xrightarrow{Haus} L$ and $(L_{i_l}) \xrightarrow{Haus} L$. Hence, as A and B are each H-closed we obtain that $L \in A \cap B \subseteq A \cup B$. We deduce that $A \cup B$ is H-closed.

Finally we verify that arbitrary intersections of H-closed sets are H-closed. Let $\{A_\alpha\}_\alpha$ be a family of H-closed sets and let $(L_i) \in (\bigcap_\alpha A_\alpha)^\infty$ such that $(L_i) \xrightarrow{Haus} L$ for some $L \in \mathcal{L}$. Then, for each α, we have that $(L_i) \in (A_\alpha)^\infty$ and $(L_i) \xrightarrow{Haus} L$, so $L \in A_\alpha$. Hence $L \in \bigcap_\alpha A_\alpha$, so $\bigcap_\alpha A_\alpha$ is H-closed. $\square$

To justify the "naturalness" of the natural topology, we begin by showing that it induces continuity of the join map. Assume throughout that the subset $(\mathbb{R}^2 \times \mathbb{R}^2)_*$ of $\mathbb{R}^2 \times \mathbb{R}^2$ is endowed with the subspace topology.

Proposition 3.12. *Endow $\mathcal{L}$ with the natural topology H. Then the join map*

$$\vee : (\mathbb{R}^2 \times \mathbb{R}^2)_* \to \mathcal{L}, \ (\mathbf{p}, \mathbf{q}) \mapsto \mathbf{p} \vee \mathbf{q}$$

is continuous.

Proof. Let $(\mathbf{p}, \mathbf{q}) \in (\mathbb{R}^2 \times \mathbb{R}^2)_*$ and let U be an open neighbourhood of $L := \vee(\mathbf{p}, \mathbf{q})$. Choose open neighbourhood bases

$$\{V_i := B(\mathbf{p}, \tfrac{1}{m_i}) : i \in \mathbb{N}, m_{i+1} \geq m_i \text{ for each } i\}$$

and

$$\{W_i := B(\mathbf{q}, \tfrac{1}{n_i}) : i \in \mathbb{N}, n_{i+1} \geq n_i \text{ for each } i\}$$

of $\mathbf{p}$ and $\mathbf{q}$, respectively, so that $V_i \cap W_i = \varnothing$ for each i. Now, suppose that $\vee(V_i \times W_i) = V_i \vee W_i$ is not contained in U for all i. Then we may choose a sequence $\big((\mathbf{p}_i, \mathbf{q}_i)\big) \in \big((\mathbb{R}^2 \times \mathbb{R}^2)_*\big)^\infty$ such that for each i: $(\mathbf{p}_i, \mathbf{q}_i) \in V_i \times W_i$ and $\vee(\mathbf{p}_i, \mathbf{q}_i) \in \vee(V_i \times W_i)\backslash U$. By construction we have that $\big((\mathbf{p}_i, \mathbf{q}_i)\big) \to (\mathbf{p}, \mathbf{q})$ and hence, by Proposition 3.9, $(\mathbf{p}_i \vee \mathbf{q}_i) \to \mathbf{p} \vee \mathbf{q}$. But $\mathcal{L}\backslash U$ is closed, so this implies that $L = \mathbf{p} \vee \mathbf{q} \in \mathcal{L}\backslash U$ — contradicting our choice of U. Thus there is an open subset $V_i \times W_i$ of $(\mathbb{R}^2 \times \mathbb{R}^2)_*$ for some i such that $\vee(V_i \times W_i) \subseteq U$ and hence $(\mathbf{p}, \mathbf{q}) \in V_i \times W_i \subseteq \vee^{-1}(U)$. This shows that $\vee^{-1}(U)$ is open; we deduce that $\vee$ is continuous. □

3.4 Characterisations of the Natural Topology

The intersection map $\wedge$ is said to be *stable* with respect to some topology on $\mathcal{L}$ if $\wedge$ is continuous and its domain $\mathcal{L}^2_*$ is open in the product space $\mathcal{L}^2$. With the aim of proving the stability of $\wedge$ and the uniqueness of the natural topology, we establish some other ways to characterise the open sets of the natural topology. To do so we define various topologies on $\mathcal{L}$ and show they are equivalent to H.

For disjoint subsets A and B of $\mathbb{R}^2$, the set $A \times B$ lies in $(\mathbb{R}^2 \times \mathbb{R}^2)_*$; we adopt the notation

$$A \vee B := \vee(A, B) = \{\mathbf{p} \vee \mathbf{q} : \mathbf{p} \in A,\ \mathbf{q} \in B\}.$$

Define four topologies on $\mathcal{L}$ as follows.

(F) *Finest topology*: the finest (i.e., largest) topology on $\mathcal{L}$ such that $\vee$ is continuous.

(OJ) *Open Join topology*: the topology on $\mathcal{L}$ generated by the subbasis (cf. [1], Chapter 2.2)

$$\mathfrak{B}_{\mathsf{OJ}} = \big\{U_1 \vee U_2 : U_1,\ U_2 \text{ are open disjoint subsets of } \mathbb{R}^2\big\}.$$

(IJ) *Interval Join topology*: the topology on $\mathcal{L}$ generated by the subbasis

$$\mathfrak{B}_{\mathsf{IJ}} = \big\{I_1 \vee I_2 : I_1,\ I_2 \text{ are } \textit{opposite} \text{ open intervals}\big\},$$

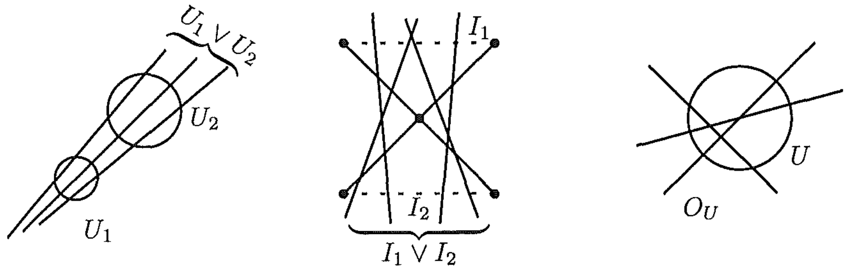
where open (or closed) intervals $]\mathbf{a}, \mathbf{b}[$ and $]\mathbf{c}, \mathbf{d}[$ are said to be *opposite* (with respect to a point $\mathbf{p}$) if the lines $\mathbf{a} \vee \mathbf{d}$ and $\mathbf{b} \vee \mathbf{c}$ are distinct and intersect at the point $\mathbf{p} \in]\mathbf{a}, \mathbf{d}[\wedge]\mathbf{b}, \mathbf{c}[$.

(OM) *Open Meet topology*: the topology on $\mathcal{L}$ generated by the subbasis

$$\mathfrak{B}_{\mathsf{OM}} = \big\{O_U : U \text{ is an open subset of } \mathbb{R}^2\big\},$$

where for each open $U \subseteq \mathbb{R}^2$,

$$O_U := \{L \in \mathcal{L} : L \cap U \neq \varnothing\}.$$



(a) OJ (b) IJ (c) OM

Figure 3.10: Topologies on $\mathcal{L}$

Theorem 3.13. *The topologies* H, F, OJ, IJ *and* IJ *are equivalent.*

Proof. We show that $\mathsf{OJ} \subseteq \mathsf{OM} \subseteq \mathsf{H} \subseteq \mathsf{F} \subseteq \mathsf{IJ} \subseteq \mathsf{OJ}$.
• $\mathsf{OJ} \subseteq \mathsf{OM}$: Letting $U_1 \vee U_2 \in \mathfrak{B}_{\mathsf{OM}}$, we show that $U_1 \vee U_2 \subseteq O_{U_1} \cap O_{U_2}$. Let $\mathbf{p} \vee \mathbf{q} \in U_1 \vee U_2$. Then the line $\mathbf{p} \vee \mathbf{q}$ intersects the open subsets U_1 and U_2 of $\mathbb{R}^2$ at $\mathbf{p}$ and $\mathbf{q}$, respectively, so we have that

$$\mathbf{p} \vee \mathbf{q} \in O_{U_1} \cap O_{U_2}.$$

Hence $U_1 \vee U_2 \subseteq O_{U_1} \cap O_{U_2}$, and thus finite intersections and arbitrary unions of elements of $\mathfrak{B}_{\mathsf{OJ}}$ lie in the collection of finite intersections and arbitrary unions of elements of $\mathfrak{B}_{\mathsf{OM}}$. That is, $\mathsf{OJ} \subseteq \mathsf{OM}$.
• $\mathsf{OM} \subseteq \mathsf{H}$: It suffices to show that for each open $U \subseteq \mathbb{R}^2$, the set O_U lies in H, since this implies that finite intersections and arbitrary unions of elements of $\mathfrak{B}_{\mathsf{OM}}$ lie in H, *i.e.*, $\mathsf{OM} \subseteq \mathsf{H}$. With this in mind suppose that

$$\mathcal{L} \backslash O_U = \{L \in \mathcal{L} : L \cap U = \varnothing\}$$

is *not* H-closed. Then there is a sequence $(L_i) \in (\mathcal{L}\backslash O_U)^\infty$ and a line $L \in O_U$ (*i.e.*, $L \notin \mathcal{L}\backslash O_U$) such that $L_i \xrightarrow{Haus} L$. Let $\mathbf{p} \in L \cap U \neq \varnothing$; then, since $\mathbf{p} \in L$, there is a sequence $(\mathbf{p}_i) \in (L_i)$ converging to $\mathbf{p}$. Now, since U is open in $\mathbb{R}^2$ there is a $\delta > 0$ such that if $\mathbf{q} \in \mathbb{R}^2$ and $||\mathbf{p} - \mathbf{q}|| < \delta$, then $\mathbf{q} \in U$. Given this δ, using $(\mathbf{p}_i) \to \mathbf{p}$, there is an $N > 0$ such that $||\mathbf{p}_i - \mathbf{p}|| < \delta$ for all $i > N$; thus, for all $i > N$, we have that $\mathbf{p}_i \in U$ and so $\mathbf{p}_i \in L_i \cap U$ — contradicting that $L_i \cap U = \varnothing$ for each i. We deduce that $\mathcal{L}\backslash O_U$ is H-closed and hence $O_U \in \mathsf{H}$.
• $\mathsf{H} \subseteq \mathsf{F}$: Since $\vee$ is continuous with respect to the topology H (Proposition 3.12) we have that $\mathsf{H} \subseteq \mathsf{F}$.
• $\mathsf{F} \subseteq \mathsf{IJ}$: It suffices to show that $\mathfrak{B}_{\mathsf{IJ}}$ is a basis for F, since then an element of F is the union of elements of $\mathfrak{B}_{\mathsf{IJ}}$ and hence lies in IJ. Let $V \in \mathsf{F}$ and let $L = \mathbf{p} \vee \mathbf{q} \in V$. By the definition of F,

$$\vee^{-1}(V) = \left\{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^2 \times \mathbb{R}^2)_* : \mathbf{x} \vee \mathbf{y} \in V\right\}$$

is open in $(\mathbb{R}^2 \times \mathbb{R}^2)_*$ and hence open in $\mathbb{R}^2 \times \mathbb{R}^2$. Let D be an open ball in $\vee^{-1}(U) \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ with centre $(\mathbf{p}, \mathbf{q})$. Choose points $(\mathbf{p}_1, \mathbf{q}_1)$, $(\mathbf{p}_2, \mathbf{q}_2)$ in D such that $\{\mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{q}_1, \mathbf{q}_2\}$ lie on opposite sides of $\mathbf{p} \vee \mathbf{q}$ in $\mathbb{R}^2$. Then $I_1 =]\mathbf{p}_1, \mathbf{q}_1[$ and $I_2 =]\mathbf{p}_2, \mathbf{q}_2[$ are opposite open intervals and we have

$$\mathbf{p} \vee \mathbf{q} \in I_1 \vee I_2 \subseteq V.$$

This shows that $\mathfrak{B}_{\mathsf{IJ}}$ is a basis for F.
• $\mathsf{IJ} \subseteq \mathsf{OJ}$: As for the $\mathsf{OM} \subseteq \mathsf{H}$ case, it suffices to show that each subbasic set $I_1 \vee I_2$ of IJ lies in OJ. Let $I_1 =]\mathbf{p}_1, \mathbf{q}_1[$ and $I_2 =]\mathbf{p}_2, \mathbf{q}_2[$ be opposite (with respect to $(\mathbf{p}_1 \vee \mathbf{q}_2) \wedge (\mathbf{p}_2 \vee \mathbf{q}_1)$) open intervals for some points $\mathbf{p}_i, \mathbf{q}_i$, $i \in \{1, 2\}$. Let U_1, U_2 and U_3 be the components of

$$\mathbb{R}^2 \backslash \big((\mathbf{p}_1 \vee \mathbf{p}_2) \cup (\mathbf{p}_2 \vee \mathbf{q}_2) \cup (\mathbf{q}_2 \vee \mathbf{q}_1) \cup (\mathbf{q}_2 \vee \mathbf{p}_1)\big)$$

as shown in Figure 3.11.

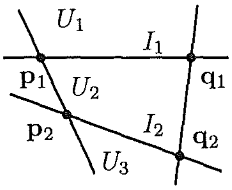


Figure 3.11

Let $\mathbf{a} \in I_1$, $\mathbf{b} \in I_2$. Since I_1 and I_2 are open, there are open balls $D_1 \subseteq I_1$ and $D_2 \subseteq I_2$ centred on $\mathbf{a}$ and $\mathbf{b}$, respectively. As U_1 and U_2 are open in $\mathbb{R}^2$, the open line segments $D_1 \cap (\mathbf{a} \vee \mathbf{b})$ and $D_2 \cap (\mathbf{a} \vee \mathbf{b})$ intersect U_1 and U_2 at some points $\mathbf{a}'$ and $\mathbf{b}'$, respectively. Hence $\mathbf{a} \vee \mathbf{b} = \mathbf{a}' \vee \mathbf{b}' \in U_1 \vee U_2$. Conversely, by the convexity of $\langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_2, \mathbf{q}_1 \rangle$, a line $\mathbf{a} \vee \mathbf{b} \in U_1 \vee U_3$ must intersect each of the open intervals I_1 and I_2 precisely once at some points $\mathbf{a}'$ and $\mathbf{b}'$, respectively. Hence $\mathbf{a} \vee \mathbf{b} = \mathbf{a}' \vee \mathbf{b}' \in U_1 \vee U_3$, so $U_1 \vee U_3 \subseteq I_1 \cup I_2$. Thus, $I_1 \vee I_2 = U_1 \vee U_3 \in \mathsf{OJ}$. □

We proceed to use the interval join-characterisation (IJ) of the natural topology to prove that it induces a stable intersection map.

Proposition 3.14. *The set $\mathcal{L}^2_*$ is open in $\mathcal{L}^2$ with respect to the product topology on* H.

Proof. Let $(K, L) \in \mathcal{L}^2_*$ and let $\mathbf{a}_i, i = 1, \ldots, 4$, be points in the four different open quarter-planes determined by K and L, chosen so that $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle$ is convex (see Figure 3.12). Set $I_i =]\mathbf{a}_i, \mathbf{a}_{i+1}[$ for $1 \le i \le 4$, where $\mathbf{a}_5 = \mathbf{a}_1$. Then I_2 and I_4 are opposite intervals (with respect to $K \wedge L$), as are I_1 and I_3, so $I_2 \vee I_4$ and $I_1 \vee I_3$ are open in $\mathcal{L}$ with respect to $\mathsf{IJ} = \mathsf{H}$. Put $Q := \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle$, let $L_1 \in I_1 \vee I_3$ and $L_2 \in I_2 \vee I_4$; then $L_1 \cap Q = [\mathbf{x}_1, \mathbf{x}_3]$ and $L_2 \cap Q = [\mathbf{x}_2, \mathbf{x}_4]$ for some $\mathbf{x}_i \in]\mathbf{a}_i, \mathbf{a}_{i+1}[$, $1 \le i \le 4$. Thus, by Lemma 3.5, $[\mathbf{x}_1, \mathbf{x}_3]$ and $[\mathbf{x}_2, \mathbf{x}_4]$, and hence L_1 and L_2, intersect, so $(L_1, L_2) \in \mathcal{L}^2_*$. Whence we have

$$(K, L) \in (I_2 \vee I_4) \times (I_1 \vee I_3) \subseteq \mathcal{L}^2_*,$$

and $\mathcal{L}^2_*$ is open in $\mathcal{L}^2$.

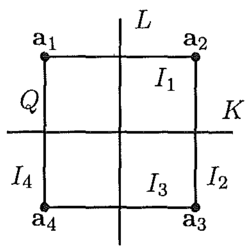


Figure 3.12

□

Proposition 3.15. *The intersection map $\wedge : \mathcal{L}^2_* \to \mathbb{R}^2 : (K, L) \mapsto K \wedge L$ is continuous.*

Proof. Let U be an open subset of $\mathbb{R}^2$ and let $(K, L) \in \wedge^{-1}(U)$. Letting $\mathbf{p} = K \wedge L \in U$, by Lemma 3.7 there is a convex quadrangle $Q = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle$ such that $\mathbf{p} \in \mathrm{int}(Q)$. Choose points $\mathbf{b}_i \in \mathrm{int}(Q)$, for $1 \le i \le 4$, so that each $\mathbf{b}_i$ lies in a different open quarter-plane determined by K and L. Let $I_i =]\mathbf{b}_i, \mathbf{b}_{i+1}[$ for $1 \le i \le 4$ (with $\mathbf{b}_5 = \mathbf{b}_1$). Since

$$\{L_1 \wedge L_2 : L_1 \in I_2 \vee I_4,\ L_2 \in I_1 \vee I_3\} \subseteq U,$$

we have

$$(K, L) \in (I_2 \vee I_4) \times (I_1 \vee I_3) \subseteq \wedge^{-1}(U).$$

This shows that $\wedge^{-1}(U)$ is open in $\mathcal{L}^2_*$, so $\wedge$ is continuous. □

We conclude by showing that the natural topology is the unique.

Theorem 3.16. *Let* $(\mathbb{R}^2, \mathcal{L}, \mathbf{F})$ *be a Moulton plane. Then the natural topology* H *on* $\mathcal{L}$ *is the only topology such that* $\mathcal{L}^2_*$ *is open in the product space* $\mathcal{L}^2$ *and both the join and intersection maps,*

$$\begin{aligned} \vee : (\mathbb{R}^2 \times \mathbb{R}^2)_* \to \mathcal{L} : (\mathbf{p}, \mathbf{q}) \mapsto \mathbf{p} \vee \mathbf{q}, \\ \wedge : \mathcal{L}^2_* \to \mathbb{R}^2 : (K, L) \mapsto K \wedge L, \end{aligned}$$

respectively, are continuous.

Proof. By Propositions 3.12, 3.14 and 3.15 we know that H is indeed a topology on $\mathcal{L}$ for which the results of the theorem statement hold. Let X be some topology on $\mathcal{L}$ such that $\vee$ and $\wedge$ are continuous and $\mathcal{L}^2_*$ is open in the product space $\mathcal{L}^2$. We show that $\mathsf{X} = \mathsf{H}$, thus showing that H is indeed unique. Since $\mathsf{H} = \mathsf{F}$ (Theorem 3.13) is the finest topology on $\mathcal{L}$ such that $\wedge$ is continuous, we have that $\mathsf{X} \subseteq \mathsf{H}$.

Conversely, we show that each subbasis element of $\mathsf{OM} = \mathsf{H}$ is contained in X, so that since OM is the collection of all finite intersections and arbitrary unions of subbasis elements, we obtain that $\mathsf{OM} \subseteq \mathsf{X}$. Let $O_U = \{L \in \mathcal{L} : L \cap U \neq \varnothing\}$ be a subbasic set in OM, for some open $U \subseteq \mathbb{R}^2$.

Define the map $\pi_1 : \mathcal{L}^2_* \to \mathcal{L}$ by $(K, L) \mapsto K$. We first verify that π_1 is an open map. Let V be an open subset of $\mathcal{L}^2_*$, which by the hypothesis is open in $\mathcal{L}^2$ with respect to the product topology on X, so V is open in $\mathcal{L}^2$ as well. Now, by the definition of the product space, for each $(K, L) \in V$ there are X-open subsets V_1 and V_2 of $\mathcal{L}$ such that $(K, L) \in V_1 \times V_2 \subseteq V$. Thus

$$\pi_1(K, L) \in V_1 = \pi(V_1 \times V_2) \subseteq \pi_1(V).$$

That is, for each $\pi_1(K, L) \in \pi_1(V)$, there is an X-open subset V_1 such that $\pi_1(K, L) \in V_1 \subseteq \pi_1(V)$, so $\pi_1(V)$ is X-open in $\mathcal{L}$. Therefore, since $\wedge^{-1}(U)$ is open in $\mathcal{L}^2_*$ by the continuity of $\wedge$, we see that

$$\begin{aligned} O_U &= \pi_1\big(\{(K, L) \in \mathcal{L}^2_* : K \cap U \neq \varnothing\}\big) \\ &= \pi_1\big(\{(K, L) \in \mathcal{L}^2_* : K \wedge L \in U\}\big) \\ &= \pi_1\big(\wedge^{-1}(U)\big) \end{aligned}$$

is X-open in $\mathcal{L}$. That is, $O_U \in \mathsf{X}$ and we have that $\mathsf{OM} \subseteq \mathsf{X}$. □

4 Conclusion

This project has succeeded in providing an introduction to an interesting means by which topological spaces may arise — that is, by imposing continuity conditions on the geometric operations of incidence structures. By having looked at just two examples, the potential complexity of such structures is quite evident. We have, however, merely scratched the surface of a fascinating field: given an incidence structure with continuity conditions, one's task is to ascertain the structure of the resulting topology. The project has thus inspired the author to pursue further research in the subject.

References

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